Dynamics and Pure Nash Equilibria in Human Decisions

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Mestrado em Matemâtica
Departamento de Matemâtica
2016

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Todas as correções determinadas pelo júri, e só essas, foram efetuadas.

O Presidente do Júri,

Porto, / /

Acknowledgements

I would like to express my gratitude to my supervisor Maria Victoria Otero Espinar for the useful help, remarks and for the support through the learning process of this master thesis. Furthermore I would like to thank my co-supervisor Alberto Adrego Pinto for introducing me to the topic as well as for the warm welcome that he and his family gave me in Portugal.

Also, I would like to thank my family for supporting me throughout the entire process and for giving me the needed support. Specially, I would like to thank my girlfriend Dominika for her infinite patience, support and love. This work is devoted to all of them.

Resumo

Nesta tese, revemos algumas definições e resultados relativos a Teoria de Jogos Evolutiva, em particular a equação do replicador e sua versão generalizada, o replicador polimatricial. Revemos o método que estuda o comportamento assimptótico dinâmico do replicador polimatricial. Revemos o modelo de decisão com dois tipos de decisões e dois tipos de indivíduos. A seguir generalizamos este modelo para o caso de k indivíduos e n decisões. O EDO associado ao modelo de decisão é um replicador polimatricial, o que poderá permitir uma aplicação futura do método de comportamento dinâmico assimptótico para provar a existência de caos no EDO associado ao modelo de decisão. Para finalizar, encontramos condições suficientes e necessárias para a existência de equilíbrios de Nash puros no modelo de decisão no caso 2×2, que poderão ser generalizadas para o caso $k \times 2$.

Palavras-chave

Teoria de jogos, Teoria de Jogos Evolutivos, equilíbrio de Nash puro, dinâmica assimptótica.

Abstract

In this thesis, we review some definitions and results concerning Evolutionary Game Theory, with special attention to the replicator equation and its generalized version, the polymatrix equation. Related with this equation, we review a method that studies the asymptotic behaviour near to the boundary of the polymatrix equation. After, we review the decision model consisting of two types of individuals and two decisions. Then, we generalize this model to the case of k types of individuals and n decisions. The ODE associated to the decision model is indeed a polymatrix equation, which might permit in the future the application of the assymptotic dynamical method to prove the existence of chaos in the ODE associated to the decision model. To end, we find necessary and sufficient conditions for the existence of pure Nash equilibrium in the decision model with 2 decisions and 2 types of individuals, that can be extended to the $k \times 2$ case.

Key words

Game Theory, Evolutionary Game Theory, pure Nash equilibria, asymptotic dynamics.

Contents

Introduction

Game Theory was originally developed as a theory of human strategic behaviour based on rational decision making. The book *Theory of Games and Economic Behavior* [15], published in 1944 by John von Neumann and Oskar Morgenstern, is considered as the initial point of this theory.

Evolutionary Game Theory (EGT) was started in 1973 with a paper written by Maynard Smith and Price [14], although there were some previous and isolated attempts to mix Game Theory and Ecology. Evolutionary Game Theory consists of the application of Classical Game Theory to Evolutionary Biology, field which is based on the idea that the genes of an organism determine its observable characteristics. Those organisms which have better genes, will have better characteristics and, consequently will produce more offspring. As a result, the best genes will tend to win over the time. EGT relies on the idea of Darwinian natural selection. The three basic principles of natural selection can be stated as: existence of heritable variation, struggle for existence and influence of heritable variation in the struggle (see [4]). The first one refers to the fact that the children must resemble their parents but with some mutations. The second is based on the consideration of some limiting factors that force populations to compete for the existing resources. The last one means that some genetic characteristics will beat others during the struggle. EGT concerns about the interaction of organisms in a population, according to the three stated principles of natural selection. The success of an individual in the struggle will depend on its genetic characteristics and fitness is the measure of how good are that characteristics. As the success depends on how individuals interact with others, fitness cannot be measured in isolation but in the context of the population. With this idea in mind, we can think of an analogy from this situation with a Game Theory conflict. The organisms gene characteristics and behaviours are the strategies from the game, fitness corresponds to the payoffs for the individuals and the fitness of a certain individual depends on the characteristics (strategies) of the other individuals with which interacts.

Ecology and EGT are closely related. As an example, it serves that the most common models for populations (Lotka-Volterra equations) and the most common models for the dynamics frequencies of strategies (replicator equations) are mathematically equivalent. The equivalence was proved in 1981 by Josef Hofbauer (see [12]). In the 1920s Lotka-Volterra equations were independently introduced by Alfred Lotka and Vito Volterra to model the evolution of chemical and biological ecosystems, respectively. This equations are now a mathematical model widely used in many fields such as Physics, Biology or Economy. In 1978, Peter Taylor and Leo Jonker introduced the replicator equation which is a central piece of EGT (see [21]). Other types of replicator-like equations are bimatrix replicator, and its generalization, the polymatrix replicator. Bimatrix replicator was introduced in 1981 to study the dynamics of bimatrix games, also called asymmetric games, where two groups of individuals within a population (e.g. males and females), interact using different sets of strategies, for example, n and m strategies, respectively. In such games there are no interactions within each group. Polymatrix games are the generalization of bimatrix game to the case where we have any finite number of groups interact, each of them using a different set of strategies. In [2], Alishah and Duarte define the polymatrix replicator, which is the generalization of the replicator equation to polymatrix games. This extended replicator equation is defined on a product of simplexes, which it will be an example of what is called polytope. This is how EGT is related with polytopes theory. In [6], a method to study the behaviour of dynamical systems defined on polytopes was introduced. It is also treated on the later works [1] and [18].

This work is organized in three chapters. In chapter 1, we give a preliminary introduction to EGT, starting with the basic concepts of the field and focusing on the mentioned replicator equations. We also remark the equivalence between this equation and Lotka-Volterra model. This chapter is intended to be a short introduction to the field, with the basic elements needed to follow the rest of the work. In the second chapter, we focus on polymatrix games. For that, we introduce the mentioned method presented by P. Duarte, with special attention to the polymatrix replicator equation. In chapter 3, we present the decision model introduced in [19], first in the 2×2 case and then we extend it to the general $k \times n$ case. We finally conclude that this general model can be seen as a polymatrix replicator and thus, the method presented in the second chapter can be used to study the behaviour of this system. In this dissertation, for the model of dimension 2×2, we give necessary and sufficient conditions for the existence of pure Nash equilibria for all point in $\mathbb{R}^2$. A future work will be the extension of this results to the $k \times 2$ case. All the figures appearing on this thesis were made using GeoGebra.

Chapter 1

Evolutionary Game Theory

This chapter is intended to give a short introduction to some of the basic concepts in Evolutionary Game Theory and to present, at first, replicator equation for symmetric games and then, the analogous versions for asymmetric games.

In the first section we give a introduction to the Classical Game Theory and some of its key concepts, such as pure and mixed strategies and Nash equilibria.

In the second section we introduce Evolutionary Game Theory and the main concept of evolutionarily stable strategy. We also see the relation between this concept and Nash equilibria.

In the third section, we let dynamics come into service, with the replicator equation, which models the evolution of the growth rate of the different strategies. We derive the equation and then we see it is invariant on its phase space, the simplex. Replicator equation in this section is valid only for symmetric games.

In the fourth section, we briefly introduce Lotka-Volterra equations, and we state its equivalence to the replicator equation.

In the fifth section, we consider asymmetric games for populations divided in two groups and its corresponding replicator equation.

To conclude, in the last section we generalize the replicator equation for asymmetric games with a finite number p of groups. We introduce some of the notation used in [1] and [18], which will be important in the rest of the work.

1.1 Classical Game Theory

Our aim here is to give a short introduction to the terminology related to the Classical Game Theory. For 'game', we understand an interactive decision problem which involves two or more individuals, which we shall call 'players', whose decisions determine their gains inside the game.

More specifically, we want to introduce static games, that is, those games in which a single decision is made by each player, and each player has no knowledge of the decision made by the other players before making its own decision. To keep the notation simple, we will only concentrate on two-player games.

In this field, the concept of strategy plays an important role. The word *strategy* is derived from the Greek word *strategos*, which means *military commander* or, colloquially, *plan of action*. Thus, a strategy is a rule for choosing a decision whenever it is needed.

If we want to describe a static game, we need to specify the following:

- the set of players, indexed by $i \in \{1, \ldots, n\}$.
- a pure strategy set, S_i, for each player.
- payoffs for each player i facing any other player j, which are given by functions $\pi_i : S_i \times S_j \to \mathbb{R}$.

As it is usual in this theory, we assume that payoffs represent the preferences of rational individuals and that the aim of the players is to maximise their payoff. A pure strategy is one strategy for which there is no randomisation. When randomisation is considered, we arrive to the concept of mixed strategy, which, for a player i, specifies the probabilities of choosing any strategy $s \in S_i$. A mixed strategy will be denoted by σ_i and the set of all possible mixed strategies for player i will be denoted by Σ_i.

So, imagine one player has to choose between the pure strategies $\{s_1, \ldots, s_n\}$. Then, a mixed strategy will be represented by a probability vector $\sigma = (p(s_1), \ldots, p(s_n))$, being $p(s_i)$ the probability to play strategy s_i. In this way, pure strategies are also represented as such vectors: $s_1 = (1, 0, 0, \dots, 0)$, $s_2 = (0, 1, 0, \dots, 0)$ and so on. Thus, mixed strategies can be written as a linear combinations of pure strategies

$$\sigma = \sum_{i=1}^{n} p(s_i) s_i.$$

If we denote the probability of using pure strategy s by $p(s)$ for player 1 and by $q(s)$ for player 2, then the payoffs for mixed strategies are given by

$$\pi_i(\sigma_1, \sigma_2) = \sum_{s_1 \in S_1} \sum_{s_2 \in S_2} p(s_1) q(s_2) \pi_i(s_1, s_2).$$

The payoffs are assumed to be a representation of the preferences of rational individuals or, depending on the case, of their biological fitness, so every individual wants to maximise its payoff.

Notation 1. A solution of a game is a pair of strategies that two rational players can use. Solutions will be denoted enclosing a strategy pair within brackets, like, for example (σ_1, σ_2), where the strategy adopted by player 1 is placed first.

Remark 1.1. As here we are considering games where all players have the same payoffs, we will note $\pi_i = \pi$, for all $i = 1, \dots, n$.

Now, we introduce one of the key concepts of Game Theory, the well-known concept of a Nash equilibrium.

Definition 1.2. A Nash equilibrium (for two player games) is a pair of strategies (σ_1^*, σ_2^*) such that:

(1) $\pi(\sigma_1^*, \sigma_2^*) \geq \pi(\sigma_1, \sigma_2^*)$, for all $\sigma_1 \in \Sigma_1$.

(2) $\pi(\sigma_1^*, \sigma_2^*) \geq \pi(\sigma_1^*, \sigma_2)$, for all $\sigma_2 \in \Sigma_2$.

The equilibrium is called symmetric when $\sigma_1^* = \sigma_2^*$.

Nash equilibrium can be interpreted as one strategy in which changing is penalized, that is, if one individual uses a Nash equilibrium strategy and then changes to another strategy, then its payoff is reduced.

Definition 1.3. The support of a strategy σ is the set $S(\sigma) \subset S$ of all strategies for which $p(s) > 0$, for the strategy σ.

Let us give one simple example to make clearer the concept of support.

Example 1.4. Let the strategy set for an individual be $S = \{L, M, R\}$ and let σ be a mixed strategy given by $\sigma = (0, p, 1-p)$, with $p \in (0,1)$. Then, $S(\sigma) = \{M, R\}$.

The following result tells us that, inside the support of a Nash equilibrium, deviation is not penalized.

Proposition 1.5. *Let (σ_1^*, σ_2^*) be a Nash equilibrium. Also, let S_1^* be the support of σ_1^* and S_2^* the support of σ_2^*. Then,*

(1) $\pi(s, \sigma_2^*) = \pi(\sigma_1^*, \sigma_2^*)$*, for* $s \in S_1^*$.

(2) $\pi(\sigma_1^*, s) = \pi(\sigma_1^*, \sigma_2^*)$*, for* $s \in S_2^*$.

We state now a well-known result which was proved by John Nash in his PhD thesis, whose proof uses fixed point results. For more details, one can check the books [8] and [16].

Theorem 1.6. *Every game that has a finite strategic form*[1] *has at least one Nash equilibrium, involving pure or mixed strategies.*

Example 1.7. Prisoner's dilemma.

Two criminals are catched by the police and they are being questioned in relation with a important case. They are in different rooms, without any connection between them, so they cannot communicate to each other. The police knows that they are guilty (at least one of them) but, without any confession, they cannot be sent to jail. The captain has an idea to make them talk. One deal is offered to both prisoners, independently: if you confess that the other is guilty and the other fails to confess, then you will be set free and your partner will be sent to jail for 10 years. If both of

[1] A game with finite strategic form is a game with finite number of players and finite number of pure strategies for each player

you confess the crime, then both of you will go to jail for 9 years. If no one confess, then both of you will be in the prison for only one year.

Let S be stay silent and C be confess. The strategic set for this game is $\{S, C\}$. Payoff for this game is given in the following table.

	S	C
S	$-1, -1$	$-10, 0$
C	$0, -10$	$-9, -9$

What is the best each prisoner can do? Start by thinking as if you were prisoner 1 (P_1). When prisoner 2 (P_2) is quiet, it is better for you to confess as you will obtain freedom, rather than 1 year in prison. On the other hand, when P_2 confesses, then you should also confess, as you will obtain 9 years in prison, rather than 10 if you stay silent. So, the best for you is to confess, independently of what P_2 will do. As this game is symmetric, for P_2 the best choice is also to confess, so, if we assume that both prisoners act rationally, then they both should confess. So, the strategy (C, C) is a Nash equilibrium for this game.

It is interesting to note that, although (C, C) is a Nash equilibrium, it is not the best situation for both-players, as they will obtain a better payoff if they confess and the other stays quiet.

Remark 1.8. Just to mention, we can also consider another version of the prisoner's dilemma, on which one prisoner knows what the other has done. In that case, the outcome of the game will change. This set up falls into the so called dynamic games, where decisions are made in different times.

1.2 Evolutionarily Stable Strategies

In the Classical Game Theory, the outcome depends on the choices made by rational individuals. So, the interpretation for Nash equilibria is that each player uses a strategy that is a best response to the strategy chosen by the other. In 1973, Maynard Smith and Price first applied the ideas of Game Theory to model the behaviour of animals (see [14]). At that time, it was odd to apply such theory, which always considered that rational players were involved, to animals that might not seem rational. To successfully adapt that theory to animals, three shifts were made in the following concepts: strategy, equilibrium and player interactions.

Now, strategic sets are genotypic variants and individuals inherit strategies, that are passing through the generations. Equilibrium now corresponds to the new concept of evolutionary stable strategy (ESS), that is just a strategy that, when all population is using it, it fares well against mutation introduced into the population. Player interactions are considered to be random, as it is chance who picks up the agents who will play their strategies only according to their genome.

The payoff to an individual using some strategy is identified with the fitness of that strategy in the current population. Fitness is measured by the expected offspring of the individual. Because animals with more fitness will leave more offspring, the population composition will change at each stage.

So the frame in this theory will be a population, whose individuals choose between a set of n pure strategies. The important thing here is the frequency of use of each strategy inside the population and how this frequency evolves with the time.

It is interesting to mark that there is some underlying dynamics in this concept, as we are interested to know how the frequencies evolve. When referring to population frequency, we are going to use the concept of population profile. Let S denote the set of possible pure strategies.

Definition 1.9. A population profile is a vector x that gives the probability $x(s)$ with which each strategy $s \in S$ is played in the population.

Given an individual using a strategy σ and inside a population with profile x, its payoff is defined by

$$\pi(\sigma, \mathrm{x}) = \sum_{s \in S} p(s)\pi(s, \mathrm{x}).$$

We can see this as the payoff obtained for an individual using σ and playing 'against' the population with profile x. The payoffs for such case correspond to the number of descendants that individuals have. This is called game against the field, which we shall just mention here.

In the called pairwise contest game[2], where an individual plays against a randomly selected opponent, the payoff for any individual using a strategy σ is

$$\pi(\sigma, \mathrm{x}) = \sum_{s\in S}\sum_{s'\in S} p(s)x(s')\pi(s,s').$$

This is the same payoff than the one for an individual using strategy σ facing an opponent with strategy σ', where, under σ', we have $p'(s) = x(s),$ for $s \in S$. So, it is always possible to associate a two-player game with a population game involving pairwise contests.

Let us come back now to the idea introduced earlier. For this purpose, consider a population where in the initial stage, all the individuals adopt some strategy σ^*. Suppose that a mutation occurs and a small proportion ε of individuals use some other strategy σ. The population after the appearance of the mutants is called the post-entry population and is denoted by x_ε. Evolutionary stable strategies are defined as follows.

Definition 1.10. A mixed strategy is called evolutionarily stable (E.S.S.) if, for $\sigma \neq \sigma^*$, there exists some $\bar{\varepsilon} = \bar{\varepsilon}(p) > 0$ such that

$$\pi(\sigma^*, \mathrm{x}_\varepsilon) > \pi(\sigma, \mathrm{x}_\varepsilon) \tag{1.1}$$

holds for all $0 < \varepsilon < \bar{\varepsilon}$.

That is, σ^* is the best choice when another strategy σ appears, in small amount, in the population, so σ will not spread and σ^* will remain in the population.

Proposition 1.11. *Let σ^* be an E.S.S. in a pairwise contest. Then, for all $\sigma \neq \sigma^*$ either*

(1) $\pi(\sigma^*, \sigma^*) > \pi(\sigma, \sigma^*)$, or

(2) $\pi(\sigma^*, \sigma^*) = \pi(\sigma, \sigma^*)$ and $\pi(\sigma^*, \sigma) > \pi(\sigma, \sigma)$.

[2]this name is used in the book [22].

Conversely, if either (1) or (2) holds for each $\sigma \neq \sigma^*$ *in a two-player game, then* σ^* *is an E.S.S. in the corresponding population game.*

Some authors define an evolutionarily stable strategy directly using (1)-(2).

Observe that condition (1) is exactly the definition of σ^* being a Nash equilibrium. Thus, we have the following result:

Proposition 1.12. *Every evolutionarily stable strategy* σ^* *is also a Nash equilibrium.*

Proof. Note that Nash equilibrium condition in this case is $\pi(\sigma^*, \sigma^*) \geq \pi(\sigma, \sigma^*)$, for all $\sigma \neq \sigma^*$. □

It is not true that all games have an E.S.S., as we will see in **Example 1.22** where a game without any E.S.S. is considered.

1.3 Replicator dynamics

With E.S.S., we started to introduce 'some' dynamics into scene, as we were interested into the study of how the population will evolve under natural selection. Here, our purpose is to go one step further and introduce the called replicator dynamics, the point where Game Theory meets Dynamical Systems theory. We consider a population formed by different kinds of individuals that use some strategy and pass it to its descendants. Here, individuals only use pure strategies from the set $\mathbf{S} = \{s_1, \ldots, s_n\}$. We set n_i to be the number of individuals of the population that use strategy s_i. The number of individuals in the whole population is given by $N = \sum_{i=1}^{n} n_i$. It will be interesting, as it was in the case of E.S.S., to know the frequencies of use for each pure strategy, that is,

$$x_i = \frac{n_i}{N}.$$

In this way, we define

Definition 1.13. A population state is a vector $\mathrm{x} = (x_1, \ldots, x_n)$, where $x_i = \frac{n_i}{N}$.

Define the average payoff in the population by:

$$\bar{\pi}(\mathrm{x}) = \sum_{i=1}^{n} x_i \pi(s_i, \mathrm{x}),$$

where $\pi(s_i, \mathrm{x})$ denotes the payoff for an individual with strategy s_i.

Let β be the per capita birth rate and δ the per capita death rate of the population. Then, the time evolution of the number of individuals using strategy s_i is given by:

$$\dot{n}_i = (\beta - \delta + \pi(s_i, \mathrm{x}))n_i = (\beta - \delta + \pi(s_i, \mathrm{x}))x_i N.$$

Also, the rate of change of the population size is

$$\begin{aligned}\dot{N} &= \sum_{i=1}^{n} \dot{n}_i = \sum_{i=1}^{n} (\beta - \delta) n_i + \sum_{i=1}^{n} \pi(s_i, \mathrm{x}) n_i \\ &= (\beta - \delta) \sum_{i=1}^{n} n_i + N \sum_{i=1}^{n} \pi(s_i, \mathrm{x}) x_i = (\beta - \delta + \bar{\pi}(\mathrm{x}))N.\end{aligned}$$

We are mainly interested on the rate change of each x_i. From the relation $n_i = x_i N$, we have

$$\dot{n}_i = \dot{x}_i N + x_i \dot{N}$$

and

$$\dot{x}_i N = \dot{n}_i - x_i \dot{N} = (\beta - \delta + \pi(s_i, \mathrm{x}))x_i N - (\beta - \delta + \bar{\pi}(\mathrm{x}))x_i N$$

Dividing by N, we arrive to the following system of O.D.E's:

$$\dot{x_i} = (\pi(s_i, \mathrm{x}) - \bar{\pi}(\mathrm{x}))\, x_i, \text{ for } i \in \{1, \ldots, n\}, \tag{1.2}$$

called replicator equation. Here, if one strategy has better payoff than average, will spread among the population.

The phase space of the replicator equation is the simplex Δ^{n-1}, defined as

$$\Delta^{n-1} = \left\{ x \in \mathbb{R}^n / \sum_{i=1}^{n} x_i = 1,\ x_i \geq 0 \right\}.$$

The following is satisfied:

Lemma 1.14. *The simplex Δ^{n-1} and its faces are invariant under* (1.2).

Proof. The $n-$plane given by the condition $\sum_{i=1}^{n} x_i = 1$, which contains Δ^{n-1}, is invariant, as:

$$\left(\sum_{i=1}^{n} x_i\right)^{\cdot} = \sum_{i=1}^{n} \dot{x_i} = \sum_{i=1}^{n} x_i \pi(s_i, \mathrm{x}) - \sum_{i=1}^{n} x_i \bar{\pi}(\mathrm{x}) = \bar{\pi}(\mathrm{x}) - \bar{\pi}(\mathrm{x}) \sum_{i=1}^{n} x_i$$
$$= \bar{\pi}(\mathrm{x}) - \bar{\pi}(\mathrm{x}) = 0.$$

Also, given any $q-$dimensional face of the simplex, the $q-$plane containing that face is also invariant, with $0 < q < n$. □

There exist some relations between the equilibria of (1.2) and E.S.S. and Nash equilibria strategies, that we shall see for pairwise contest games. We give a definition and introduce some notation before continuing.

Definition 1.15. Let $\dot{\mathrm{x}} = f(\mathrm{x})$ be a dynamical system and x^* be an equilibrium of the system. The fixed point x^* is said to be:

a) Stable: if for all neighbourhood V of x^*, there is a neighbourhood $V_1 \subset V$ such that every solution $x(t)$ with $x(0) = x_0 \in V_1$ is defined and lies in V for all $t > 0$.

b) Unstable: if it is not stable.

c) Asymptotically stable: if it is stable and, further, V_1 can be chosen so that $x(t) \to \mathrm{x}^*$, as $t \to \infty$.

Notation 2. Let $\mathbf{F}$ be the set of fixed points and let $\mathbf{A}$ be the set of asymptotically stable fixed points of the replicator equation (1.2). Consider now the symmetric game corresponding to the replicator dynamics and let $\mathbf{N}$ be the set of Nash equilibrium strategies and let $\mathbf{E}$ be the set of E.S.S.s in that game.

It holds that, for any pairwise contest game, the following relationships are satisfied for any strategy σ and the corresponding population state x:

(1) If $\sigma \in \mathbf{E}$, then $\mathrm{x} \in \mathbf{A}$.

(2) If $\mathrm{x} \in \mathbf{A}$, then $\sigma \in \mathbf{N}$.

(3) If $\sigma \in \mathbf{N}$, then $\mathrm{x} \in \mathbf{F}$

If we identify strategies with their corresponding population states, we can express this as the following chain:

$$\mathbf{E} \subset \mathbf{A} \subset \mathbf{N} \subset \mathbf{F}.$$

Our aim now is to state that fact, dividing it in three theorems. Before that, we introduce the concept of relative entropy function, which has something in common with the stability of fixed points:

Definition 1.16. Given a population state $\mathrm{x}^* = (x_1^*, x_2^*, \ldots, x_n^*)$, the relative entropy function is defined as

$$\begin{aligned} V : \mathbb{R}^n &\longrightarrow \mathbb{R} \\ \mathrm{x} &\longmapsto V(\mathrm{x}) = -\sum_{i=1}^{k} x_i^* log\left(\frac{x_i}{x_i^*}\right) \end{aligned}$$

The importance of this function resides on the fact that if its time derivative, evaluated in points along solution trajectories and with x^* fixed point, is positive, the fixed point x^* is unstable and. In the case it is zero, then the evolution of the population is periodic around the fixed point.

It is also important in this theory, as it usually serves as a Lyapunov function for the replicator equation.

We recall the concept of Lyapunov function:

Definition 1.17. Let $\dot{\mathrm{x}} = f(\mathrm{x})$ be a dynamical system with a fixed point at x^*. A strict Lyapunov function for the dynamical system is a scalar function $V(\mathrm{x})$ defined in a neighbourhood of the fixed point x^* and satisfying the conditions

(i) $V(\mathrm{x}^*) = 0$

(ii) $V(\mathrm{x}) > 0$ for $\mathrm{x} \neq \mathrm{x}^*$

(iii) $\dot{V}(\mathrm{x}) < 0$ for $\mathrm{x} \neq \mathrm{x}^*$.

If such a function exists for the fixed point x^*, then x^* will be asymptotically stable. This fact is useful for the proof of the following result, where the relative entropy function serves as a Lyapunov function. (see [22]).

Theorem 1.18. *Given a strategy σ which is an E.S.S., the population with $\mathrm{x} = \sigma$ is asymptotically stable.*

To complete the previous subset chain we state the following results, whose proofs can be found in [22].

Theorem 1.19. *If x is an asymptotically stable fixed point of the replicator dynamics, then the symmetric strategy pair (σ, σ) with $\sigma = \mathrm{x}$ is a Nash equilibrium.*

Theorem 1.20. *If (σ, σ) is a symmetric Nash equilibrium, then the population state $\sigma = \mathrm{x}$ is a fixed point of the replicator dynamics.*

We finish this section with one example. Before that, we give a necessary definition (see [9]).

Definition 1.21. Let $\dot{\mathrm{x}} = f(\mathrm{x})$ be a dynamical system. A heteroclinic orbit for the system is a union of distinct fixed points and the trajectories connecting them. Heteroclinic cycles are closed paths formed of heteroclinic orbits.

Example 1.22. Consider the well-known rock-scissors-paper game. Let us remember how it works. Two players are involved, each player has three possible strategies, rock (R), paper (P) and scissors (S). When both players choose the same strategy, no one wins and, in other case, R beats S, S beats P and P beats R. A payoff table for this game can be:

	R	S	P
R	$0,0$	$1,-1$	$-1,1$
S	$-1,1$	$0,0$	$1,-1$
P	$1,-1$	$-1,1$	$0,0$

The game has an only Nash equilibrium (σ, σ), where $\sigma = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. Nevertheless, this strategy is not an E.S.S. We have that

$$\pi(\sigma, R) = \frac{1}{3}\pi(R,R) + \frac{1}{3}\pi(P,R) + \frac{1}{3}\pi(S,R) = 0 - \frac{1}{3} + \frac{1}{3} = 0$$

and

$$\pi(R,R) = 0,$$

so, the strategy (σ, σ) cannot be an E.S.S.

Being Nash equilibrium is a necessary condition to be E.S.S., thus, no such strategy exists for this game.

Now, let us make the replicator dynamics into scene. Let x_1 be the proportion of $R-$players, x_2 the proportion of $S-$ players and x_3 the proportion of $P-$players. The population state x is given by $\mathrm{x} = (x_1, x_2, x_3)$. We have:

$$\pi(S, \mathrm{X}) = x_1\pi(S,S) + x_2\pi(S,R) + x_3\pi(S,P) = x_3 - x_2,$$

$$\pi(R, \mathrm{X}) = x_1\pi(R,S) + x_2\pi(R,R) + x_3\pi(R,P) = x_1 - x_3,$$

$$\pi(P, \mathrm{X}) = x_1\pi(P,S) + x_2\pi(P,R) + x_3\pi(P,P) = x_2 - x_1,$$

so, the replicator equation in this case results:

$$\dot{x}_1 = x_1(x_3 - x_2)$$
$$\dot{x}_2 = x_2(x_1 - x_3)$$
$$\dot{x}_3 = x_3(x_2 - x_1)$$

The fixed points for this system are $(1,0,0) \equiv R, (0,1,0) \equiv S, (0,0,1) \equiv P$ and $\mathrm{z} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.

Also, it can be seen that, restricting ourselves to the boundaries, that the behaviour is $R \to S \to P \to R$. This implies that there exists some kind of cyclic behaviour around the other fixed point z. To know whether this spiral cycles are periodic orbits, attracting to the boundaries or to the fixed point z will be possible with the observation made about the derivative of the relative entropy function.

The relative entropy function is here given by

$$V(\mathrm{x}) = -\sum_{i=1}^{3} \frac{1}{3} log(3x_i)$$

and its derivative is

$$\frac{d}{dt}V(\mathrm{x}) = -\frac{1}{3}\frac{\dot{x}_i}{x_i} = -\frac{1}{3}\left((x_3 - x_2) + (x_1 - x_3) + (x_2 - x_1)\right) = 0.$$

Then, we must have periodic orbits around z.

In [23], the Lyapunov function $V(\mathrm{x}) = x_1x_2x_3$ is suggested. As $\frac{d}{dt}V(\mathrm{x}) = 0$, then the solution curves are the level curves of the function V, which are closed curves surrounding z.

Another approach to detect whether the boundary heteroclinic cycles are attracting or not is given in [11], but we shall not introduce it here.

Remark 1.23. Of special interest is the case when the function $\pi(s_i, \mathrm{x})$ is linear. For this case, there exists a matrix $A = (a_{ij}) \in \mathbb{M}_n(\mathbb{R})$ such that $\pi(s_i, \mathrm{x}) = (A\mathrm{x})_i$. Also, we have that $\bar{\pi}(\mathrm{x}) = \mathrm{x}^T A\mathrm{x}$. The replicator equation can be written as:

$$\dot{x_i} = \left((A\mathrm{x})_i - \mathrm{x}^T A\mathrm{x}\right) x_i, \text{ for } i \in \{1, \ldots, n\}. \tag{1.3}$$

1.4 Lotka-Volterra equation

In the 1920s, Alfred Lotka suggested that a system of two biological species could oscillate permanently. For that, he purposed the following system of differential equations:

$$\begin{aligned} \frac{dx}{dt} &= ax - bxy \\ \frac{dy}{dt} &= -cy + dxy, \end{aligned} \tag{1.4}$$

where $x(t)$ and $y(t)$ denoted the population number of both species and where $a, b, c, d > 0$. He published this equation (among others) in a book entitled *Elements of Physical Biology* in 1925. However, it did not draw much attention at that time. Nevertheless, the famous mathematician Vito Volterra rediscovered the same model independently, while studying a fishery problem. He was interested into explaining the reason why the proportion of cartilaginous fish was increased during the First World War period. These fishes are predators of smaller fish and, during this period, fishing effort was reduce, so that could be the reason why cartilaginous fish proportion raised. To explain this situation, he proposed the model (1.4), where now $x(t)$ stands for the number of prey and $y(t)$ stands for the number of predators. He published it in an article in 1926. Although Lotka discovered the model earlier, his work would not always be mentioned.

The general Lotka-Volterra equation for n populations of competing species has the form

$$\dot{x}_i = x_i \left(r_i + \sum_{j=1}^{n} a_{ij} x_j \right) \quad i = 1, \ldots, n, \tag{1.5}$$

where x_i denotes the population densities, r_i are the intrinsic growth or decay rate from population i and the a_{ij} describe the effect of population j over population i. Matrix $A = (a_{ij})$ is called interaction matrix. The state space of (1.5) is the orthant

$$\mathbb{R}^n_+ = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n / \; x_i \geq 0 \text{ for } i = 1, \ldots, n\}.$$

Boundary points from the state space correspond to coordinate planes $x_i = 0$, which means that species i is absent. Thus, the boundary is invariant, as if $x_i(t) = 0$ is the solution of the $i-$th equation of (1.5) satisfying $x_i(0) = 0$. As a consequence, $\mathbb{R}^n_+$ is invariant under (1.5).

After this short introduction of Lotka-Volterra equation, let us state its equivalence to the replicator equation (1.3).

Note that Lotka-Volterra equation is a quadratic equation on $\mathbb{R}^n_+$, while replicator equation is cubic on the compact Δ^{n-1}. However, it turns out that the replicator equation in n variables $x_1, \ldots, x_n$ is equivalent to the Lotka-Volterra equation in

$n-1$ variables $y_1, \ldots, y_{n-1}$. We state here this result, proved by Hofbauer, whose proof can be checked in [13].

Theorem 1.24. *Let $\hat{S}^{n-1} = \{x \in \Delta^{n-1} /\ x_n > 0\}$. There exists a differentiable and invertible map*

$$F : \hat{S}^{n-1} \longrightarrow \mathbb{R}_{+}^{n-1}$$

mapping the orbits of the replicator equation

$$\dot{x_i} = \left((A\mathrm{x})_i - \mathrm{x}^T A\mathrm{x}\right) x_i, \quad i = 1, \ldots, n$$

onto the orbits of the Lotka-Volterra equation

$$\dot{y_i} = y_i \left(r_i + \sum_{j=1}^{n} a'_{ij} y_j \right), \quad i = 1, \ldots, n-1,$$

where $r_i = a_{in} - a_{nn}$ and $a'_{ij} = a_{ij} - a_{nj}$.

As a consequence, results about replicator equation can be carried over to the Lotka-Volterra equations.

1.5 Bimatrix games

Up to this point, we have only considered situations where all players are in symmetric positions, that is, they have same payoffs and same set of strategies. However, in many conflicts that is not the case. There can be some differences between the players, depending on some characteristics, as being male or female, be weak or strong, etc. To solve this kind of situations, for pairwise conflicts, we are led to bimatrix games.

We distinguish between players in position I and position II. First player has n strategies and the second has m strategies. Payoffs for this players are given by matrices A and B, respectively. We assume that players in position I can only play against players in position II and vice-versa. Mixed strategies are denoted by $p \in \Delta^{n-1}$ for player I and by $q \in \Delta^{m-1}$ for player II.

Nash equilibrium can be also defined for this kind of games.

Definition 1.25. A Nash equilibrium is a pair of strategies $(\hat{p}, \hat{q}) \in \Delta^{n-1} \times \Delta^{m-1}$ such that:

(1) $\pi_1(\hat{p}, \hat{q}) \geq \pi_1(p, \hat{q})$, for all $p \in \Delta^{n-1}$.

(2) $\pi_2(\hat{p}, \hat{q}) \geq \pi_2(\hat{p}, q)$, for all $q \in \Delta^{m-1}$.

For games in the previous sections, we had a symmetric scenario: player in position I using i against a player II using j obtains the same payoff as a player II playing j against a player I using i. For those games, we had $A = B$.

While Nash equilibria can be extended to asymmetric games, there is no obvious extension to the concept of evolutionary stability to asymmetric games.

Now, let us denote by $\mathrm{x} \in \Delta^{n-1}$ and $\mathrm{y} \in \Delta^{m-1}$ the frequencies of strategies for players I and II. We can also associate a differential equation to this asymmetric case, just as in the symmetric case, making the assumption that the rate of increase of any strategy x_i, that is, $\dot{x}_i/x_i$ is equal to the difference between its payoff, given by $(A\mathrm{y})_i$ and the average payoff in the population, given by $\mathrm{x}A\mathrm{y}$. Making the same assumption for player II strategies, we obtain two different equations:

$$\begin{aligned} \dot{x}_i &= x_i\left((A\mathrm{y})_i - \mathrm{x}A\mathrm{y}\right), \quad i = 1, \dots, n \\ \dot{y}_j &= y_j\left((B\mathrm{x})_j - \mathrm{y}B\mathrm{x}\right), \quad j = 1, \dots, m, \end{aligned} \tag{1.6}$$

which are invariant on $\Delta^{n-1} \times \Delta^{m-1}$.

1.6 Polymatrix games

Let us now introduce polymatrix games, where the population is divided in a finite number of groups, say p, each one with a finite number of strategies and where interactions between any two players are allowed (even if they are from the same group). Polymatrix games serve as a generalization of symmetric and asymmetric games that we have seen so far in this work. The differential equation associated to this game, the polymatrix replicator, which was introduced in [2], is a generalization of the replicator equations in the symmetric case (1.3) and the asymmetric case (1.6). This equation will be defined on a product of simplices.

Let us consider a population divided in p groups, which are tagged by an integer $\alpha \in \{1, \ldots, p\}$. The individuals from a given group α have a number n_α of possible strategies. This strategies are tagged by integers in the following range:

$$n_1 + n_2 + \ldots + n_{p-1} < j \leq n_1 + \ldots n_p \tag{1.7}$$

That is, any strategy is given by an integer $j = 1, \ldots, n$, with $n = n_1 + \ldots + n_p$. We will write $j \in \alpha$ to mean that j is a strategy from the group α, that is, when (1.7) occurs.

With this ingredients, we can now define what is a polymatrix game:

Definition 1.26. A polymatrix game is an ordered pair, denoted by $(\underline{n}, A)$, where $\underline{n} = (n_1, \ldots, n_p)$ is a list of positive integers, called the game type, that represents the number of decisions of each group α and where $A \in \mathbb{M}_n(\mathbb{R})$, called the payoff matrix of the game.

Given two strategies $i \in \alpha$ and $j \in \beta$, the enter a_{ij} on the matrix A represents the average payoff for an individual using strategy i who is facing another individual using strategy j. For this reason, the matrix A can be decomposed in p^2 block matrices $A^{\alpha\beta}$, of dimension $n_\alpha \times n_\beta$ and whose entries are denoted by $a_{ij} = a_{ij}^{\alpha\beta}$, with $\alpha, \beta \in \{1, \ldots, p\}$. Each block $A^{\alpha\beta}$ gives us the payoff for interactions of individuals in the group α with individuals in β.

Note that one point $x = (x^\alpha)_\alpha \in \mathbb{R}^n$ describes the state of the population in such game. The point x belongs to the following product of simplexes:

$$\Gamma_{\underline{n}} := \Delta^{n_1 - 1} \times \Delta^{n_2 - 1} \times \ldots \times \Delta^{n_p - 1} \subset \mathbb{R}^n,$$

called prism. We recall that the $n_\alpha-$dimension simplex is defined by

$$\Delta^{n_\alpha - 1} = \left\{ x \in \mathbb{R}^{n_\alpha} / \sum_{i=1}^{n_\alpha} x_i = 1,\ x_i \geq 0 \right\}.$$

The entry $x_j = x_j^\alpha$ of x represents the frequency of usage of the strategy j inside the group α. This means that we can also decompose $x \in \mathbb{R}^n$ in p parts: $x^\alpha \in \mathbb{R}^{n_\alpha}$, where $\alpha \in \{1, \ldots, p\}$.

With $(Ax)_i = \sum_{j=1}^{n} a_{ij}x_j$ we represent the payoff for a player who is choosing the strategy i and $\sum_{\beta=1}^{p}(x^\alpha)^t A^{\alpha\beta}x^\beta$ gives us the payoff that a player can expect in the game. Thus the difference $(Ax)_i - \sum_{\beta=1}^{p}(x^\alpha)^t A^{\alpha\beta}x^\beta$ represents the relative fitness of the strategy $i \in \alpha$, inside the group α. We obtain the polymatrix replicator dynamics as:

$$\frac{dx_i^\alpha}{dt} = x_i^\alpha \left((Ax)_i - \sum_{\beta=1}^{p}(x^\alpha)^t A^{\alpha\beta}x^\beta \right), \tag{1.8}$$

$\forall i \in \alpha$, with $\alpha \in \{1, \ldots, p\}$.

This system can be also represented as

$$\frac{\dot{x}_i^\alpha}{x_i^\alpha} = (Ax)_i - \sum_{\beta=1}^{p}(x^\alpha)^t A^{\alpha\beta}x^\beta. \tag{1.9}$$

The phase space of this O.D.E's system is the prism $\Gamma_{\underline{n}}$ defined before. We will denote the flow associated to (1.8) by $X_A = X_{(\underline{n},A)}$.

Remark 1.27. Both notations (1.8) and (1.9) are the same. To justify the presence of both, we say that the first one appears in the works of Duarte and Peixe ([1], [18]) while the latter is more usual in the rest of the literature and, in particular, it will be used later on in this work.

Remark 1.28. When $p = 1$, equation (1.9) is just the usual replicator equation associated to the payoff matrix A and defined on the space $\Gamma_{\underline{n}} := \Delta^{n-1}$.

For $p = 2$, with the restriction $A^{11} = A^{22} = 0$ (that is, individuals who are from the same group cannot compete), equation (1.9) becomes the bimatrix replicator equation, associated to the payoff matrices A^{12} and A^{21}, defined on $\Gamma_{\underline{n}} := \Delta^{n_1-1} \times \Delta^{n_2-1}$.

Lemma 1.29. *The flow X_A leaves invariant the prism $\Gamma_{\underline{n}}$. In particular, it leaves each of the simplexes $\Delta^{n_\alpha-1}$ invariant, for $\alpha \in \{1, \ldots, p\}$.*

Proof. Similar to the proof of **Lemma 1.14**. $\square$

Now we will set when two polymatrix games with the same type $\underline{n}$ are equivalent.

Definition 1.30. Two polymatrix games $(\underline{n}, A)$ and $(\underline{n}, B)$ are equivalent if the rows of the block matrix $A^{\alpha\beta} - B^{\alpha\beta}$ are equal, for $\alpha, \beta \in \{1, \ldots, p\}$. We will indicate this by $(\underline{n}, A) \sim (\underline{n}, B)$.

Now we will state two results concerning the vector field X_A. The first one, motivates the previous definition, as two equivalent games define the same flow, while the second characterizes the equilibria of the polimatrix replicator.

Proposition 1.31. *Let $(\underline{n}, A)$ and $(\underline{n}, B)$ be two equivalent polymatrix games. Then, $X_{(\underline{n},A)} = X_{(\underline{n},B)}$ on the prism $\Gamma_{\underline{n}}$.*

Proposition 1.32. *Let $(\underline{n}, A)$ be a polymatrix game and $q \in$ int$(\Gamma_{\underline{n}})$. The point q is an equilibrium of* (1.8) *if and only if $(Aq)_i = (Aq)_j$, for all strategies $i, j \in \alpha$ and for $\alpha \in \{1, \ldots, p\}$.*

Example 1.33. (From [18]) Consider the polymatrix game $((5), A)$, with

$$A = \begin{pmatrix} 0 & -2 & 2 & -2 & 2 \\ 2 & 0 & -2 & 0 & 0 \\ -2 & 2 & 0 & -3 & 0 \\ 2 & 0 & 3 & 0 & -2 \\ -2 & 0 & 0 & 2 & 0 \end{pmatrix}.$$

That is, we have a population formed by only one group, whose members can decide between 5 possible decisions. The point $q = \left(\frac{1}{8}, \frac{5}{16}, \frac{1}{8}, \frac{1}{8}, \frac{5}{16}\right)$ satisfies $(Aq)_1 = (Aq)_2 = (Aq)_3 = (Aq)_4 = (Aq)_5$ and then, q is an equilibrium of the polymatrix replicator derived from $((5), A)$.

References

The introduction of this thesis was elaborated mainly following the references [4], [7], [10] [18].

We have used for the introduction to E.S.S. and replicator dynamics, mainly, the book of James Webb [22] as its exposition serves for an introduction to the field and it is also a source of examples. For this part, he have also checked the book of Hofbauer and Sigmund [13], which was used mainly for the Lotka-Volterra equation and bimatrix games. For an introduction, more orientated to the biological applications, a reference could be the book of Nowak [17].

For the polymatrix replicator section, we basically checked two sources, the PhD thesis of Telmo Peixe [18] and the article [1].

The book by Nicolas Bacaër [3] is a great source to know the history behind the Lotka-Volterra equation.

Chapter 2

Dynamics of the polymatrix replicator

Duarte and Peixe had already shown the existence of chaos using their method, for two particular examples (check [18] for an example of a polymatrix replicator and [5] for a Lotka Volterra system). In the future, we would like to use the present method to classify the possible asymptotic dynamics of our model and, in particular, to show the existence of chaos for some values.

The idea we pursue in this chapter is to introduce succinctly the method appearing in [1], [18], which was firstly presented in [6].

The method envelopes the asymptotic dynamics of the flow associated to vector fields defined in certain geometrical objects, called polytopes, which will be defined more precisely later. The process is the following: start with a vector field X defined on a polytope, then consider its associated flow φ_X^t. From certain local data obtained from X near each vertex of the polytope, we construct a piecewise linear vector field, which inhabits in a new object, called dual cone of the initial polytope. Through the study of this latter vector field, we are able to analyse the asymptotic behaviour of φ_X^t along the heteroclinic network of the polytope. To make this study, we use Poincaré maps for the original vector field and for the piecewise constant field.

The idea of this chapter is to introduce briefly the terminology used in the mentioned sources ([1], [18] and [6]) to set up the basis of a future work.

In the first section, we start introducing the called polytopes. This geometrical object is the natural place where many replicator equations are defined. In the second, we introduce vector fields defined on polytopes and we focus our attention in the vector field defined by the polymatrix equation. The characters, which are the eigenvalues on the pure strategies, are computed for this equation. In the third section, we introduce a change of coordinates which sends any vector field defined on a polytope to a piecewise constant vector field defined on a new object, called dual cone of the original polytope, which is introduced in the fourth section.

In the fifth section, the piecewise linear fields we obtained are studied. We associate a graph to the vertices of the polytope that have a certain kind of behaviour with respect to the field, that ensures the existence of heteroclinic cycles in the edges of the polytope. From this graph, we are interested on the existing cycles and on the called structural sets, which are the minimal set of edges through which every cycle should pass. Next, Poincaré maps are defined to study when the flow returns to cross sections glued to the structural set.

In the next section, Poincaré maps for the piecewise constant flow are given. It is stated that, asymptotically, this two kinds of Poincaré maps are in some sense equivalent.

In the last section, a brief idea about the projetive Poincaré map is given.

2.1 Polytopes

Definition 2.1. A simple d-dimensional polytope is a compact convex subset $\Gamma^d \subset \mathbb{R}^n$ of dimension d and affine support E_d for which there exist a family of affine functions $\{f_i : E_d \to \mathbb{R}\}_{i\in I}$ such that

i) $\Gamma^d = \cap_{i\in I} f_i^{-1}([0, +\infty))$.

ii) $\Gamma^d \cap f_i^{-1}(0) \neq \emptyset$, for all $i \in I$.

iii) given any $J \subset I$ such that $\Gamma^d \cup \left(\cap_{j \in J} f_j^{-1}(0) \right) \neq \emptyset$, then the the linear 1-forms df_j are linearly independent at every point $p \in \cap_{j \in J} f_j^{-1}(0)$.

We will say that $\{f_i\}_{i \in I}$ is the defining family of the polytope Γ^d.

Now we will see more precisely what we understand for face of a polytope.

Definition 2.2. Let Γ^d be a simple polytope with defining family $\{f_i\}_{i \in I}$. A non-empty subset $\rho \subset \Gamma^d$ is called a $r-$face if there exists $d - r$ functions, $f_{i_1}, \ldots, f_{i_{d-r}}$ in $\{f_i\}_{i \in I}$ such that we can obtain ρ as

$$\rho = \Gamma^d \cap f_{i_1}^{-1}(0) \cap \ldots \cap f_{i_{d-r}}^{-1}(0).$$

$K^r(\Gamma^d)$ will denote the set of all $r-$faces of Γ^d.

Each element in $V := K^0(\Gamma^d)$ is called vertex of Γ^d, each element in $E := K^1(\Gamma^d)$ is called edge of Γ^d and the $(d-1)-$faces of Γ^d, that is, elements of $F := K^{d-1}(\Gamma^d)$, are called faces of the polytope.

Remark 2.3. The term simple in the definition of polytope means that each vertex of Γ^d has exactly d incident edges.

Definition 2.4. A corner of the polytope is an element on the set

$$C := \{(v, \gamma, \sigma) \in V \times E \times F : \gamma \cap \sigma = \{v\}\}.$$

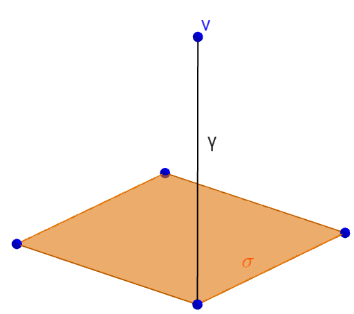

Figure 2.1: One corner (v, σ, γ) of the polytope $[0, 1]^3$

Definition 2.5. Given a vertex $v \in V$, we denote by E_v and F_v the set of all edges, respectively faces, which contain the vertex v.

Remark 2.6.
1. Given a family $\{f_i\}_{i\in I}$ defining a polyhedron Γ^d, by condition ii) in the definition of a polytope, the correspondence $i \mapsto \Gamma^d \cap f_i^{-1}(0)$ induces a one to one map between the index family I and the set of the polytope's faces F. Thus, from now on we shall assume that the family defining a polytope is always indexed in F in such a way that $\sigma = \Gamma^d \cap f_\sigma^{-1}(0)$.
2. Notice that any pair of the three elements in a corner determines uniquely the third one. So, we will refer to the corner (v, γ, σ) shortly as (v, γ) or (v, σ).
3. An edge γ with endpoints v_1 and v_2 determines two corners, (v_1, γ, σ_1) and (v_2, γ, σ_2), that are called the end corners of γ. The faces σ_1 and σ_2 are called the opposite faces of γ.
4. The sets E_v and F_v have exactly d elements, as Γ^d is a simple polytope.

We note that, because of iii) in **Definition 2.1**, for every $v \in V$ the covectors $\{(df_\sigma)_v / \, \sigma \in F_v\}$ are linearly independent. This implies that, inside a neighbourhood U_v of the vertex v, the functions $\{f_\sigma / \, \sigma \in F_v\}$ can be used to define a coordinate system for Γ^d. More precisely, given a vertex $v \in V$, define

$$\begin{aligned} \psi_v : U_v &\longrightarrow \mathbb{R}^{F_v} \equiv \mathbb{R}^d \\ q &\longmapsto (\psi_\sigma^q(q))_{\sigma\in F_v} := (f_\sigma(q))_{\sigma\in F_v}, \end{aligned} \tag{2.1}$$

where $\mathbb{R}^{F_v} := \{(x_\sigma)_{\sigma\in F_v} / \, x_\sigma \in \mathbb{R}\}$. The restriction of ψ_v to $\tilde{N}_v := U_v \cap \Gamma^d$ is a local coordinate system for the polytope, that sends the vertex v to the origin and every face σ to the hyperplane $\{x_\sigma = 0\}$. This restriction, denoted by ψ_v, is called the local $v-$coordinate system of Γ^d. Shrinking the neighbourhoods $\tilde{N}_v$, for the different vertices v, we can make them disjoint if needed, just making smaller the neighbourhoods U_v. We assume also that $[0,1]^d \subset \psi_v(\tilde{N}_v)$. The only thing we have to do, if this is not the case, is multiply each defining function by a positive factor.

Definition 2.7. Set $N_v := \psi_v^{-1}([0,1]^d)$. Then, $\{(N_v, \psi_v)/\, v \in V\}$ defines a pairwise disjoint coordinate system, called the vertex coordinates of Γ^d.

Example 2.8. The $n-1$ simplex, defined by

$$\Delta^{n-1} = \left\{ x \in \mathbb{R}^n / \sum_{i=1}^{n} x_i = 1,\; x_j \geq 0,\; \text{for } j \in \{1, \ldots, n\} \right\},$$

is a $n-$dimensional polytope. Its vertexes are $e_1, \ldots, e_n$, the elements of the canonical basis of $\mathbb{R}^n$. This polytope is an example of what is called canonical representation in [1]. Thus, its defining family is given by $f_i(x) = x_i$.

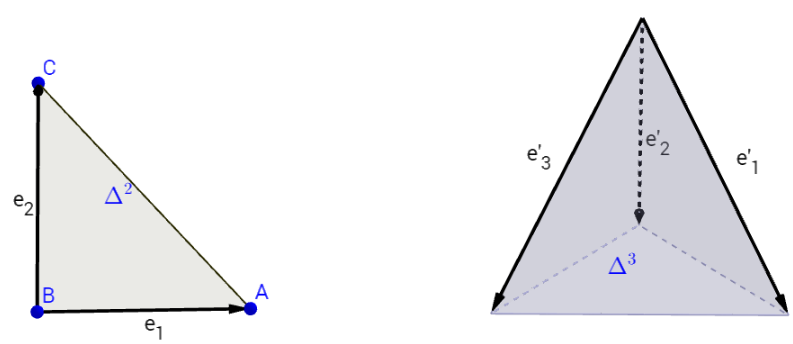

Figure 2.2: Simplexes Δ^2 and Δ^3 with its vertexes labelled.

Example 2.9. The square $[0,1]^2$ is a $2-$polytope. The defining family $\{f_i\}_{i\in I}$, $f_i : \mathbb{R}^2 \to \mathbb{R}$, $I = \{1,2,3,4\}$, for this polytope is given by:

$$f_1(x,y) = y$$

$$f_2(x,y) = 1 - x$$

$$f_3(x,y) = 1 - y$$

$$f_4(x,y) = x$$

See figure 2.3 to see the labels of the edges.

Let us compute, for this example, the coordinate system ψ_v. Let us call $v_{i,j}$ the vertex between σ_i and σ_j and $U_{v_{i,j}}$ a neighbourhood of that vertex. Then, by the definition of coordinate system (2.1), we have:

$$\psi_{v_{1,2}}(x,y) = (f_1(x,y), f_2(x,y)) = (y, 1-x), \text{ for } (x,y) \in U_{v_{1,2}},$$

$$\psi_{v_{2,3}}(x,y) = (f_2(x,y), f_3(x,y)) = (1-x, 1-y), \text{ for } (x,y) \in U_{v_{2,3}},$$

$$\psi_{v_{3,4}}(x,y) = (f_3(x,y), f_4(x,y)) = (1-y, x), \text{ for } (x,y) \in U_{v_{3,4}},$$

$$\psi_{v_{4,1}}(x,y) = (f_4(x,y), f_1(x,y)) = (x, y), \text{ for } (x,y) \in U_{v_{4,1}}.$$

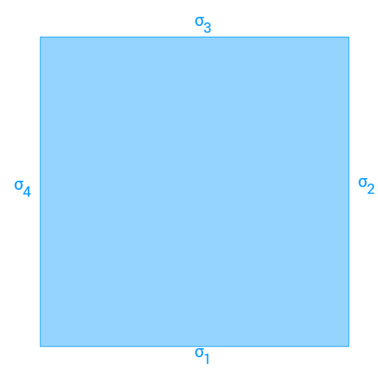

Figure 2.3: The square $[0,1]^2$.

2.2 Vector fields on polytopes

Given Γ^d a simple polytope, we are going to introduce some notation:

Notation 3. Denote by:

- $\mathcal{C}^\omega(\Gamma^d)$ the space of functions defined on Γ^d that can be extended analytically to a neighbourhood of Γ^d.

- $\mathfrak{X}^\omega(\Gamma^d)$ the space of vector fields X on Γ^d that can be extended analytically to a neighbourhood of Γ^d and that are tangent to each $r-$dimensional face of the polytope Γ^d, with $0 \leq r < d$.

In this method, analyticity is assumed just for simplicity and because the vector fields obtained by the models of the evolutionary game theory are all analytic. However, the following results can be extended to smooth vector fields.

Now, take any $X \in \mathfrak{X}^\omega(\Gamma^d)$. For each $\sigma \in F$, we have that $df_\sigma(X) = 0$ in the face $\sigma = \{q \in \Gamma^d / \, f_\sigma(q) = 0\}$[1]. Then, or $df_\sigma(X) \equiv 0$ or else $df_\sigma(X) = f_\sigma H_\sigma$, for some non identically zero function $H_\sigma \in \mathcal{C}^\omega(\Gamma^d)$.

In this last case, the vector field X is called nondegenerate when, for all faces $\sigma \in F$, the function H_σ is non identically zero on σ.

Given $v \in V$ a vertex of the polytope Γ^d, $T_v\Gamma^d$ denotes the linear space of tangent vectors to Γ^d at v.

Note that, for every corner (v, γ, σ) there is a unique vector $e_{(v,\sigma)}$ parallel to γ such that $(df_\sigma)_v(e_{(v,\sigma)}) = 1$. For $X \in \mathfrak{X}^\omega(\Gamma^d)$ the vectors $e_{(v,\sigma)}$ are eigenvectors of the derivative $(DX)_v$. Then, $H_\sigma(v)$ is the eigenvalue associated to $e_{(v,\sigma)}$ and is given by:

$$H_\sigma(v) = (df_\sigma)_v(DX)_v(e_{(v,\sigma)}).$$

Now, we define a concept which will be very important through this chapter.

Definition 2.10. Given a vector field $X \in \mathfrak{X}^\omega(\Gamma^d)$, the skeleton character of X is defined by $\chi = (\chi^v_\sigma)_{(v,\sigma)\in V\times F}$, where

$$\chi^v_\sigma := \begin{cases} -H_\sigma(v), & \text{if } \sigma \in F_v, \\ 0, & \text{otherwise.} \end{cases}$$

Polymatrix skeleton

Now, our aim is to obtain explicit expressions for the skeleton character of the vector field X_A associated to the polymatrix replicator given in (1.8). We first note that $X_A \in \mathfrak{X}^\omega(\Gamma_{\underline{n}})$.

[1]This means that $(df_\sigma)_p(X(p)) = 0$, for all $p \in \Gamma^d$ such that $f(p) = 0$.

Before continuing, we introduce some needed notation:

Notation 4. Consider the prism

$$\Gamma_{\underline{n}} := \Delta^{n_1-1} \times \Delta^{n_2-1} \times \ldots \times \Delta^{n_p-1} \subset \mathbb{R}^n,$$

which is a simple $(n-p)-$dimensional polytope. Set $d = n - p$. The defining functions of $\Gamma_{\underline{n}}$ are

$$\begin{aligned} f_i : E^d &\longrightarrow \mathbb{R} \\ x &\longmapsto f_i(x) = x_i, \end{aligned}$$

with $i \in \alpha$ and $\alpha = 1, \ldots, p$.

We will write $V_{\underline{n}} = V(\Gamma_{\underline{n}})$, $E_{\underline{n}} = E(\Gamma_{\underline{n}})$, $F\underline{n} = F(\Gamma_{\underline{n}})$ and $F_{\underline{n},v} = F_v(\Gamma_{\underline{n}})$ to denote the set of vertices, edges, faces and faces containing the vertex v, of $\Gamma_{\underline{n}}$, respectively.

$\Gamma_{\underline{n}}$ has exactly $\prod_{j=1}^{p} n_j$ vertices,

$$V_{\underline{n}} = \{e_{i_1} + \ldots + e_{i_p} / \ i_\alpha \in \alpha \text{ for } \alpha = 1, \ldots, p\},$$

where the vectors e_{i_α} stand for the canonical basis of $\mathbb{R}^n$.

We label this vertexes in the set defined as:

$$J(\underline{n}) := I_1(\underline{n}) \times \ldots \times I_p(\underline{n}),$$

where $I_\alpha(\underline{n}) := [n_1 + \ldots + n_{\alpha-1} + 1, n_1 + \ldots + n_\alpha]$, for $\alpha \in \{2, \ldots, p\}$ and, by convention, $I_1(\underline{n}) = [1, n_1]$. With this notation, we have that $j \in \alpha$ if and only if $j \in I_\alpha(\underline{n})$. Each label $(j_1, \ldots, j_p) \in J(\underline{n})$ determines the vertex

$$v_{j_1,\ldots,j_p} := e_{j_1} + \ldots + e_{j_p}.$$

This polytope has also exactly n faces:

$$F_{\underline{n}} = \{\sigma_1, \ldots, \sigma_n\},$$

with $\sigma_i := \Gamma_{\underline{n}} \cap \{x_i = 0\}$, for $i \in \{1, \ldots, n\}$.

For each vertex $v = e_{i_1} + \ldots + e_{i_p}$, the set of faces containing v is:

$$F_{\underline{n},v} = \{ \ \sigma_i / \ i \in \alpha, \ i \neq i_\alpha, \ \alpha = 1, \ldots, p \ \}.$$

Now, let us fix a face $i \in F_{\underline{n},v}$. (1.9) can be rewritten as

$$\dot{x}_i = x_i \left((Ax)_i - \sum_{k\in\alpha}\sum_{j=1}^{n} a_{kj}x_k x_j \right), \quad i \in \alpha, \alpha = 1, \dots, p. \tag{2.2}$$

If we consider the Taylor expansion in the variable x_i and around zero of the right hand side in (2.2), we have

$$\dot{x}_i = A_1 x_i + A_2 x_i^2 + A_3 x_i^3,$$

where each coefficient A_l is a polynomial on the variables x_k with $j \neq i$. If $\alpha \in \{1, \dots, p\}$ denotes the group in which i is contained, then we get:

$$A_1 = \sum_{\substack{j=1\\ j\neq i}}^{n} a_{ij}x_j - \sum_{k\in\alpha\setminus\{i\}}\sum_{\substack{j=1\\ j\neq i}}^{n} a_{kj}x_k x_j,$$

$$A_2 = a_{ii} - \sum_{\substack{j=1\\ j\neq i}}^{n} a_{ij}x_j - \sum_{k\in\alpha\setminus\{i\}} a_{ki}x_k$$

and

$$A_3 = -a_{ii}.$$

The next result, whose proof can be found in [1] or [18] gives us the expression for the character of the vector field associated to the polymatrix replicator (2.2):

Proposition 2.11. *Given the polymatrix game $(\underline{n}, A)$ as in **Definition 1.26** and let X_A be the vector field associated to a polymatrix replicator* (2.2) *defined on $\Gamma_{\underline{n}}$, we have*

$$X_A \text{ is regular} \iff A_1 \text{ does not vanish identically on } \Gamma_{\underline{n}}.$$

Furthermore, whenever X_A is regular, for every vertex $v \in V_{\underline{n}}$ with label $(j_1, \dots, j_p)$, then the skeleton character of X_A is the family $\chi = (\chi_i^v)$ given by

$$\begin{cases} \sum_{\beta=1}^{p}(a_{j_\alpha j_\beta} - a_{ij_\beta}) & \text{if } \sigma_i \in F_{\underline{n},v}, \\ \\ 0 & \text{otherwise.} \end{cases} \tag{2.3}$$

2.3 Rescaling coordinates

Given a vector field X in the polytope Γ^d, our aim here is to obtain from the original vector field, another vector field which is piecewise-constant and whose behaviour gives us information about the behaviour of X.

Our purpose now is to introduce a family of rescaling coordinates $\Psi^X_{v,\varepsilon}$ around each vertex v of the polytope, which depend on a rescaling parameter $\varepsilon > 0$. That will create a constant vector field from our original field, which is defined in the polytope. We need first to define what we call sector.

Definition 2.12. Given a vertex $v \in V$, the sector at v is defined to be

$$\Pi_v = \{(u_\sigma)_{\sigma \in F} \in \mathbb{R}^F_+ / \ u_\sigma = 0, \ \forall \sigma \notin F_v\}$$

Definition 2.13. Given a vector field $X \in \mathfrak{X}^\omega(\Gamma^d)$, a vertex v and $\varepsilon > 0$, define the rescaling $v-$coordinate by:

$$\begin{aligned} \Psi^X_{v,\varepsilon} : N_v \setminus \partial\Gamma^d &\longrightarrow \Pi_v \\ q \longmapsto y &:= \begin{cases} -\varepsilon^2 log(\psi^\sigma_v(q)) & \text{if } \sigma \in F_v \\ 0 & \text{if } \sigma \notin F_v, \end{cases} \end{aligned}$$

being ψ_v the $v-$coordinate system defined earlier (see (2.1)). It results:

$$q \longmapsto y := \begin{cases} -\varepsilon^2 log(f_\sigma(q)) & \text{if } \sigma \in F_v \\ 0 & \text{if } \sigma \notin F_v, \end{cases}$$

Remark 2.14. Notice that we have the following identifications:

$$\Pi_v \equiv \mathbb{R}^{F_d}_+ \equiv \mathbb{R}^d_+.$$

Remember that the elements of Π_v are $(u_\sigma)_{\sigma \in F} \in \mathbb{R}^F_+$, satisfying certain condition. Remember also that $d = |F|$.

2.4 Dual cone of a polytope

We are going to introduce now the concept of dual cone of a polytope, which is the natural place where the rescaled vector field, obtained in the last section, lays.

We will have a first approach to this object via geometrical interpretation for, then, giving an algebraic definition that will be more suitable for the following theory.

Let Γ^d be a simple polytope and let Γ^* be its dual polytope, that is, the one obtained associating each vertex of Γ^d with faces in Γ^* and, in general, the $r-$faces of Γ^d with the $(d-r)-$faces in Γ^*.

First, let us introduce a geometrical definition of the dual cone.

Take any point O outside of the hyperplane generated by Γ^*. The dual cone of the polytope Γ^d is defined by

$$\mathcal{C}^*(\Gamma^d) := \{O + \lambda v / \ v \in \partial\Gamma^*\},$$

that is, the cone with origin O and passing through $\partial\Gamma^*$.

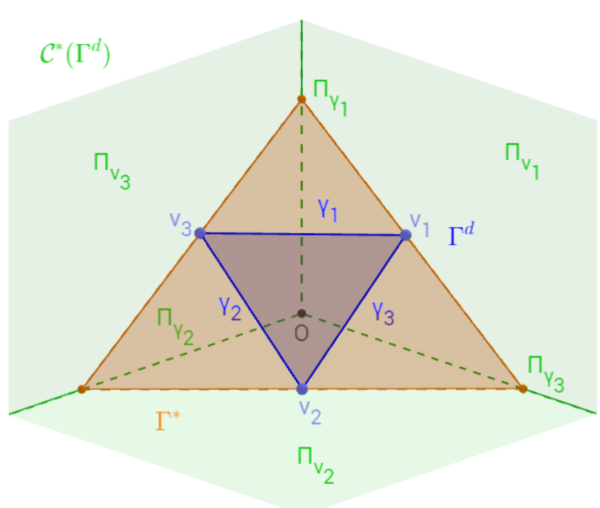

Figure 2.4: Dual cone of the polytope $\Gamma^d = \Delta^2$.

Given any face σ in Γ^d and let σ^* denote its dual face in Γ^*, we say that $\Pi_\sigma = \{O + \lambda v \ / v \in \sigma^*\}$ is a face of the dual cone.

Now, we give an alternative algebraic definition, embedding the dual cone of Γ^d into the euclidean space $\mathbb{R}^F$. This approach will be more useful for the analytic treatment. Recall that, for a vertex $v \in V$, the sector Π_v is defined as

$$\Pi_v := \{(x_\sigma)_{\sigma\in F} \in \mathbb{R}^F_+ / \ x_\sigma = 0, \ \forall \sigma \notin F_v\}.$$

Then, we define:

Definition 2.15. The dual cone of Γ^d is defined as

$$\mathcal{C}^*(\Gamma^d) = \bigcup_{v\in V} \Pi_v.$$

Looking at Figure 2.4, one can easily understand geometrically this concept. This algebraic definition is of particular interest, noting that the constant vector field we created via the rescaling coordinate lives on the dual cone.

We can extend the definition of sector from vertices to $(d-r)-$faces of Γ^d to obtain:

Definition 2.16. Given $0 \leq r \leq d$ and let $\rho \in K^{d-r}(\Gamma)$ be a $(d-r)-$face of Γ^d. The set

$$\Pi_\rho := \{(x_\sigma)_{\sigma\in F} \in \mathbb{R}^F_+ / \ x_\sigma = 0, \ \text{if } \rho \not\subset \sigma\}$$

is called $r-$dimensional face of $\mathcal{C}^*(\Gamma^d)$. The $r-$dimensional skeleton of the dual cone is defined as

$$\mathcal{C}^*_r(\Gamma^d) := \bigcup \{\Pi_\rho / \ \rho \in K^{d-r}(\Gamma^d)\}.$$

Duality plays an important role in this part, as it can be seen in the next remark.

Remark 2.17. By duality, we have the following:

a) If σ is an $r-$dimensional face of $\partial\Gamma^d$, then the face Π_σ of the dual cone $\mathcal{C}^*(\Gamma^d)$ is a $(d-r)-$dimensional sector. Also, the dual face of σ, σ^*, has dimension $d-1-r$.

b) Given faces ρ and σ of Γ^d, then $\rho \subset \sigma \Leftrightarrow \Pi_\sigma \subset \Pi_\rho$.

c) Given faces ρ and σ of Γ^d, then $\Pi_\rho \cap \Pi_\sigma = \Pi_{\rho\cap\sigma}$.

Skeleton coordinate system

Now, we introduce the skeleton coordinate system on Γ^d in the following way:

Definition 2.18. Let $W := \{x \in \mathbb{R}^F / \ 0 \le x_\sigma \le 1$ for all $\sigma \in F\}$. Then, the skeleton coordinate system on Γ^d is defined to be the map

$$\begin{aligned} \psi : \Gamma^d &\longrightarrow W \\ q &\longmapsto \psi(q) := (\psi_\sigma(q))_{\sigma\in F}, \end{aligned}$$

where $\psi_\sigma(q) := \min\{1, f_\sigma(q)\}$.

Notation 5. We will denote the element of W whose all components are equal to 1 as $\mathbb{1} = (1, \dots, 1)$.

Definition 2.19. Given a face $\sigma \in F$, define

$$N_\sigma := \{q \in \Gamma^d / \ f_\sigma(q) \le 1\}.$$

Observe that $\psi(q) = \mathbb{1}$ when $q \notin \cup_{\sigma\in F} N_\sigma$.

Definition 2.20. Given a vector field $X \in \mathfrak{X}^\omega(\Gamma^d)$ and $\varepsilon > 0$, the $\varepsilon-$rescaling Γ^d-coordinate for X is the map:

$$\begin{aligned} \Psi_\varepsilon^X : \Gamma^d \setminus \partial\Gamma^d &\longrightarrow \mathcal{C}^*(\Gamma^d) \\ q &\longmapsto \Psi_\varepsilon^X(q) := \left(-\varepsilon^2 \log(\psi_\sigma(q))\right)_{\sigma\in F}, \end{aligned}$$

where ψ_σ is given in **Definition 2.18.**

Remark 2.21. Note that the rescaling $v-$coordinate $\Psi_{v,\varepsilon}^X$, given in **Definition 2.13** is just the composition of Ψ_ε^X, restricted to N_v, with the orthogonal projection from $\mathbb{R}^F$ to Π_v.

2.5 Skeleton vector fields

Our aim now is to introduce and characterize vector fields on the dual cone. Via the rescaling coordinates, every vector field $X \in \mathfrak{X}^\omega(\Gamma^d)$ yields a piecewise constant vector field, which lives on the dual cone $\mathcal{C}^*(\Gamma^d)$. This vector field will be referred to as the skeleton vector field of X.

More precisely,

Definition 2.22. A skeleton vector field on $\mathcal{C}^*(\Gamma^d)$ is a family $\chi = (\chi^v)_{v\in V}$ of vectors in $\mathbb{R}^F$ such that each component χ^v is tangent to the face Π_v of the dual cone, for all $v \in F$.

The first example of skeleton vector field is already known. Given $X \in \mathfrak{X}^\omega(\Gamma^d)$, recall the concept of skeleton character of X, (see **Definition 2.10**). The family $\chi = (\chi^v)_{v\in V}$ obtained in this concept is a skeleton vector field. We will refer to this family as the skeleton vector field of X.

Now, we are going to study the piecewise constant flows generated by skeleton vector fields. With this idea in mind, we classify vertexes and edges of Γ^d with respect to χ. Although this definitions may seem at first strange, they are related with how the original field X behaves with the vertex or edge of the original polytope, respectively.

Definition 2.23. Given χ a skeleton vector field and a vertex v, we say that v is:

(1) $\chi-$repelling if $-\chi^v \in \Pi_v$.

(2) $\chi-$attractive if $\chi^v \in \Pi_v$.

(3) of saddle type in other case.

Remark 2.24. If χ is the skeleton vector field of $X \in \mathfrak{X}^\omega(\Gamma^d)$, then a vertex $v \in V$ is $\chi-$repelling or $\chi-$attractive if and only if v is a repelling or attractive singularity of the field X, respectively. This fact can be explained with the $v-$ rescaling coordinate system (see **Definition 2.13**).

Definition 2.25. Consider now an edge $\gamma \in E$ with end corners (v, σ) and (v', σ'). The edge γ is called $\chi-$defined if $\chi^v_\sigma \chi^{v'}_{\sigma'} \neq 0$ or $\chi^v_\sigma = \chi^{v'}_{\sigma'} = 0$.

When γ is a $\chi-$defined edge, we also say that γ is

(1) a flowing edge if $\chi_\sigma^v \chi_{\sigma'}^{v'} < 0$.

(2) an attracting edge if $\chi_\sigma^v < 0$ and $\chi_{\sigma'}^{v'} < 0$.

(3) a repelling edge if $\chi_\sigma^v > 0$ and $\chi_{\sigma'}^{v'} > 0$.

(4) a neutral edge if $\chi_\sigma^v = \chi_{\sigma'}^{v'} = 0$.

In the following, we shall focus our attention in flowing edges. Assume γ is a flowing edge such that $\chi_\sigma^v < 0$ and $\chi_{\sigma'}^{v'} > 0$. We will indicate this by $v \xrightarrow{\gamma} v'$. The vertexes v and v' are called, respectively, the source and target of γ. We will write $v = s(\gamma)$ and $v' = t(\gamma)$.

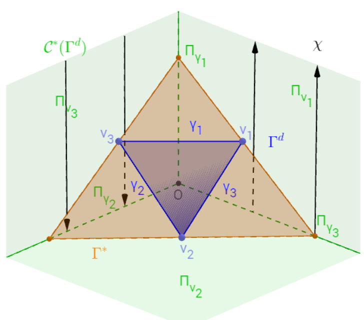

Figure 2.5: γ_1 is a neutral edge

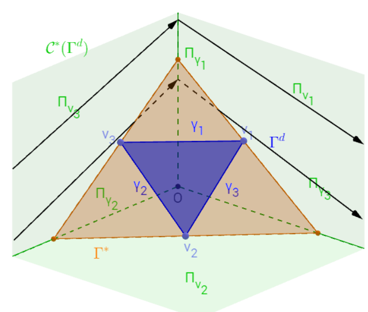

Figure 2.6: γ_1 is a flowing edge

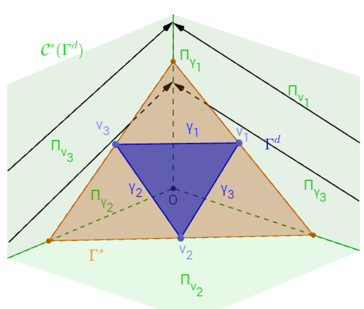

Figure 2.7: γ_1 is an attracting edge

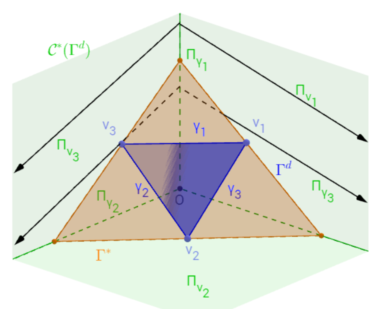

Figure 2.8: γ_1 is an repelling edge

Definition 2.26. The skeleton χ is called regular when all edges $\sigma \in E$ are $\chi-$defined.

From now on, we will assume that the skeleton vector fields which we are working with are all regular.

Let E_χ be the set of all flowing edges and recall that V is the vertex set of Γ^d. The directed graph G_χ given by $G_\chi = (V, E_\chi)$ gives us information about the existing flowing edges.

Definition 2.27. Given an interval $I \subset \mathbb{R}$, a continuous piecewise affine function $c : I \to \mathcal{C}^*(\Gamma^d)$ such that:

(1) $\dot{c}(t) = \chi^v$ if $c(t)$ is interior to some sector Π_v, with $v \in V$,

(2) the set $\{t \in I \, / c(t) \in \mathcal{C}^*_{d-1}(\Gamma^d)\}$ is finite or countable,

is called an orbit of χ.

Consider one vertex $v \in V$ and two flowing edges $v_0 \xrightarrow{\gamma} v$ and $v \xrightarrow{\gamma'} v'$. Then, there exists only one face σ' such that (v, γ', σ') is a corner of Γ^d, i.e., such that $\gamma \subset \gamma'$ and $\gamma' \cap \sigma' = \{v\}$.

Let us write down explicitly the sector defined in **Definition 2.16**, but for an edge γ:

$$\Pi_\gamma = \{(x_\sigma)_{\sigma \in F} \in \mathbb{R}^F_+ / \; x_\sigma = 0, \text{ if } \gamma \not\subset \sigma\}.$$

It will be key in the next definition, which will be related to the orbits of χ in **Proposition 2.29**.

Definition 2.28. Given $v \in V$ and two flowing edges $v_0 \xrightarrow{\gamma} v$ and $v \xrightarrow{\gamma'} v'$, we define the sector $\Pi_{\gamma,\gamma'} = \Pi^\chi_{\gamma,\gamma'}$ as

$$\Pi_{\gamma,\gamma'} := \left\{ x \in int(\Pi_\gamma) / \; x_\sigma - \frac{\chi^v_\sigma}{\chi^v_{\sigma'}} x_{\sigma'} > 0, \; \sigma \in F_v, \; \sigma \neq \sigma' \right\}$$

and the linear map $L_{\gamma,\gamma'} = L^\chi_{\gamma,\gamma'}$ by

$$\begin{aligned} L_{\gamma,\gamma'} : \Pi_{\gamma,\gamma'} &\longrightarrow \mathbb{R}^F_+ \\ x &\longmapsto \left(x_\sigma - \frac{\chi^v_\sigma}{\chi^v_{\sigma'}} x_{\sigma'} \right)_{\sigma \in F}. \end{aligned}$$

This map can be represented by the matrix

$$M_{\gamma,\gamma'} = \left(\delta_{\sigma,\sigma''} - \frac{\chi^v_\sigma}{\chi^v_{\sigma'}}\delta_{\sigma'\sigma''}\right)_{\sigma,\sigma''\in F},$$

where $\delta_{\sigma,\sigma''}$ is the Kronecker delta.

The following result relates the orbits of χ with the previous definition:

Proposition 2.29. *Given $v \in V$ and the flowing edges $v_0 \xrightarrow{\gamma} v$ and $v \xrightarrow{\gamma'} v'$, the sector $\Pi_{\gamma,\gamma'}$ is the set of points $x \in int(\Pi_\gamma)$ which are connected to the point $x' = L_{\gamma,\gamma'}(x) \in int(\Pi_{\gamma'})$ by the orbit segment $\{c(t) = x + t\chi^v / \ t \geq 0,\ c(t) \in \Pi_v\}$.*

We can also consider the same definitions for the skeleton vector field $-\chi$. We have:

Remark 2.30. The linear map $L^\chi_{\gamma,\gamma'}$ takes the sector $\Pi^\chi_{\gamma,\gamma'}$ to $\Pi^{-\chi}_{\gamma',\gamma}$ and $L^{-\chi}_{\gamma',\gamma}$ takes $\Pi^{-\chi}_{\gamma',\gamma}$ to $\Pi^\chi_{\gamma,\gamma'}$. The map $L^{-\chi}_{\gamma',\gamma}$ is the inverse of $L^\chi_{\gamma,\gamma'}$

The next remark tells us that saddle type nodes are the ones with the most interesting behaviour and also why we consider the only interior of the sectors Π_γ.

Remark 2.31. Consider two corners of the polytope, (v, γ', σ') and (v, γ'', σ'') around a vertex v. In the case that v is $\chi-$attractive or $\chi-$repelling it is not possible to connect points in $\Pi_{\gamma'}$ with points in $\Pi_{\gamma''}$ by a parallel line to the constant vector χ^v.

Furthermore, in such a case, the points from the boundary of $\Pi_{\gamma'}$ are on the intersection of more than two sectors Π_v, so, for such points, we cannot expect to have a unique orbit.

We shall not consider this type of orbits through this work.

Now, the Poincaré map associated to the flow of χ is introduced. First, consider the union of sectors $\Pi^\chi_{\gamma,\gamma'}$ for which the orbit can be constructed, denoted by

$$\Pi^\chi = \bigcup \{\Pi_{\gamma,\gamma'} / \ t(\gamma) = s(\gamma') \text{ with } (\gamma, \gamma') \in E_\chi \times E_\chi\} \subset \mathcal{C}^*_{d-1}(\Gamma^d).$$

Definition 2.32. The skeleton Poincaré map associated to χ is given by:

$$\begin{aligned} \pi^\chi : \Pi^\chi &\longrightarrow \mathcal{C}^*_{d-1}(\Gamma^d) \\ x &\longmapsto \pi^\chi(x) := L_{\gamma,\gamma'}(x), \text{ when } x \in \Pi_{\gamma,\gamma'}. \end{aligned}$$

We shall write $\pi = \pi^{\chi}$ when the skeleton vector field χ is clear from the context.

By its definition and by **Remark 2.30** it is clear that the Poincaré map associated to $-\chi$, $\pi^{-\chi}$, is the inverse of π^{χ}.

Definition 2.33. We say that a point $x \in \Pi^{\chi}$ has finite forward orbit if for some $n \in \mathbb{N}$ we have that $(\pi^{\chi})^{i}(u) \in \Pi^{\chi}$ for all $i \in \{1, \ldots, n-1\}$ and $(\pi^{\chi})^{n}(u) \notin \Pi^{\chi}$. In a similar way, we say that a point $x \in \Pi^{-\chi}$ has finite backward orbit if for some $n \in \mathbb{N}$ we have that $(\pi^{-\chi})^{i}(u) \in \Pi^{-\chi}$ for all $i \in \{1, \ldots, n-1\}$ and $(\pi^{-\chi})^{n}(u) \notin \Pi^{-\chi}$.

Otherwise, when for all $n \in \mathbb{N}$, $(\pi^{\chi})^{n}(u) \in \Pi^{\chi}$, then we say that x has infinite forward orbit and, when for all $n \in \mathbb{N}$, $(\pi^{-\chi})^{n}(u) \notin \Pi^{-\chi}$, then x has infinite backward orbit.

Definition 2.34. The maximal invariant sets are defined by

$$\Lambda^{+}(\chi) = \bigcap_{n \geq 0} (\pi^{\chi})^{-n}(\Pi^{\chi})$$
$$\Lambda^{-}(\chi) = \bigcap_{n \geq 0} (\pi^{\chi})^{-n}(\Pi^{-\chi}).$$

This sets are made of by, respectively, the points in $\mathcal{C}^{*}_{d-1}(\Gamma^{d})$ with infinite forward orbit and infinite backward orbit. The set defined by

$$\Lambda^{0}(\chi) = \Lambda^{+}(\chi) \cap \Lambda^{-}(\chi)$$

consist of points with infinite backward and forward orbit.

Definition 2.35. A sequence of edges $\xi = (\gamma_0, \ldots, \gamma_m)$ is called $\chi-$path when ξ is a path of the directed graph G_{χ}, that is:

(1) $\gamma_j \in E_{\chi}$, for $j \in \{0, 1, \ldots, m\}$.

(2) $t(\gamma_{j-1}) = s(\gamma_j)$, for $j \in \{1, \ldots, m\}$.

We say that ξ is an $\chi-$cycle when the path ξ is a cycle of G_{χ}, that is, when $\gamma_0 = \gamma_m$.

The integer m is the length of the path ξ.

Definition 2.36. An orbit segment for the Poincaré map π^χ is a finite sequence $\underline{x} = (x_0, x_1, \ldots, x_m)$, with $x_j \in \Pi^\chi$ and $x_j = \pi(x_{j-1})$ for $j \in \{1, \ldots, m\}$.

The only $\chi-$path $\xi = (\gamma_0, \ldots, \gamma_m)$ for which $x_j \in \Pi_{\gamma_{j-1},\gamma_j}$, for $j \in \{1, \ldots, m\}$ is called itinerary of $\underline{x}$.

We remark that, by definition, all the edges of an itinerary must be flowing edges.

Now, we will consider Poincaré maps along $\chi-$paths:

Definition 2.37. Given a $\chi-$path $\xi = (\gamma_0, \ldots, \gamma_m)$, define the sector Π_ξ as:

$$\Pi_\xi = \left\{x \in int(\Pi_{\gamma_0}) / \ \pi^j(x) \in int(\Pi_{\gamma_j}), \ \forall j \in \{1, \ldots, m\}\right\}.$$

The skeleton Poincaré map of χ along ξ is defined as the following composition:

$$\begin{aligned} \pi_\xi : \Pi_\xi &\longrightarrow \Pi_{\gamma_m} \\ x &\longmapsto \pi_\xi(x) := L_{\gamma_{m-1},\gamma_m} \circ \ldots \circ L_{\gamma_0,\gamma_1}(x) \end{aligned}$$

The sector Π_ξ can be rewritten as:

$$\Pi_\xi = int(\Pi_{\gamma_0}) \cap \bigcap_{j=1}^{m} (L_{\gamma_{j-1},\gamma_j} \circ \ldots \circ L_{\gamma_0,\gamma_1})^{-1} \left(int(\Pi_{\gamma_j})\right).$$

Remark 2.38. (1) Given a $\chi-$path of length one, $\xi = (\gamma, \gamma')$, then $\Pi^\chi = \Pi_{\gamma,\gamma'}$.

(2) Given two $\chi-$paths, ξ, ξ', if $\xi \neq \xi'$, then $\Pi_\xi \cap \Pi_{\xi'} = \emptyset$.

(3) Given ξ', ξ'' two $\chi-$paths such that the ending edge of ξ' coincide with the starting edge of ξ'', then we can construct a new path ξ resulting of the concatenation of $\xi'$and ξ'' and such that $\pi_\xi = \pi_{\xi''} \circ \pi_{\xi'}$.

Now we will introduce the called structural sets and the Poincaré maps associated to this sets.

Definition 2.39. A non-empty set of flowing edges $S \subset E_\chi$ is called structural set for χ if:

(1) Any possible $\chi-$cycle contains at least one edge from S.

(2) If there exists another set S' satisfying (1), then $S \subset S'$.

In general, more than one structural set for χ can exist, that is, S does not have to be unique.

Definition 2.40. A $\chi-$path $\xi = (\gamma_0, \ldots, \gamma_m)$ is called a branch of the structural set S or, shortly, $S-$branch, if:

(1) $\gamma_0, \gamma_m \in S$.

(2) $\gamma_j \notin S$, for all $j \in \{1, \ldots, m-1\}$.

The set of all $S-$branches will be denoted by $\mathscr{B}_S(\chi)$.

Definition 2.41. Define the sector $\Pi_S = \bigcup_{\xi \in \mathscr{B}_S(\chi)} \Pi_\xi$. The $S-$Poincaré map is defined as

$$\begin{aligned} \pi_S : \Pi_S &\longrightarrow \Pi_S \\ u &\longmapsto \pi_S(u) := \pi_\xi(u), \text{ when } u \in \Pi_\xi. \end{aligned}$$

Next, sufficient conditions for the dynamics of π_S to be non-trivial are given:

Proposition 2.42. *Let $X \in \mathfrak{X}^\omega(\Gamma^d)$ a vector field with associated skeleton vector field χ and with structural set $S \subset E_\chi$. If the following conditions are satisfied:*

(1) χ is regular,

(2) χ has no attracting or repelling edges,

(3) all vertices are of saddle type,

then, the set $\Lambda^0(\chi) \cap \Pi_S$ has full $(d-1)-$Lebesgue[2] measure in $\bigcup_{\gamma \in S} \Pi_\gamma$. That is, the complementary of $\Lambda^0(\chi) \cap \Pi_S$ in $\bigcup_{\gamma \in S} \Pi_\gamma$ has zero $(d-1)-$Lebesgue measure.

[2]When we write $(d-1)-$Lebesgue measure we emphasize the fact that $\Lambda^0(\chi) \cap \Pi_S$ has dimension $d-1$

Example 2.43. Coming back to the polymatrix game given in **Example 1.33**, let us illustrate with it some of the concepts introduced in this section. We denote by X_A the field generated by the polymatrix replicator derived of $((5), A)$ and χ denotes its skeleton vector field. Let us label the vertices of $\Gamma_{(5)} = \Delta^4$ by $i \in \{1, \ldots, 5\}$ to indicate the vertex e_i, from the canonical basis of $\mathbb{R}^5$. Also, represent by $\gamma = (i, j)$ the edge going from vertex i to vertex j. We shall notate this edges in the following way:

$$\gamma_1 = (1,2) \ \ \gamma_2 = (3,1) \ \ \gamma_3 = (1,4) \ \ \gamma_4 = (5,1) \ \ \gamma_5 = (2,3)$$

$$\gamma_6 = (2,4) \ \ \gamma_7 = (2,5) \ \ \gamma_8 = (3,4) \ \ \gamma_9 = (3,5) \ \ \gamma_{10} = (4,5).$$

In this case, not all edges are flowing. More precisely, edges γ_6, γ_7 and γ_9 are neutral edges.

The graph associated to the skeleton vector field χ is represented in figure 2.9, where we have also represented the structural set $S = \{\gamma_1, \gamma_4\}$.

The $S-$branches of χ are given by $\xi_1, \ldots, \xi_5$, with

$$\xi_1 = (\gamma_1, \gamma_5, \gamma_2, \gamma_1), \quad \xi_2 = (\gamma_1, \gamma_5, \gamma_8, \gamma_{10}, \gamma_1)$$

$$\xi_3 = (\gamma_1, \gamma_5, \gamma_2, \gamma_3, \gamma_{10}, \gamma_4), \quad \xi_4 = (\gamma_4, \gamma_1), \quad \xi_5 = (\gamma_4, \gamma_3, \gamma_{10}, \gamma_4)$$

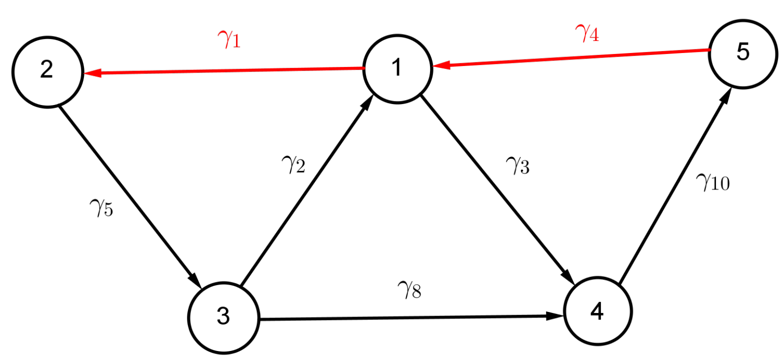

Figure 2.9: The oriented graph for χ. The structural set S is represented in red.

2.6 Asymptotic Poincaré maps

Now, we will introduce the Poincaré maps associated to vector fields $X \in \mathfrak{X}^\omega(\Gamma^d)$.

Recall that, by **Definition 2.26**, a skeleton vector field χ is regular if all edges are $\chi-$defined. We will define what we understand for regular vector fields in Γ^d:

Definition 2.44. A vector field $X \in \mathfrak{X}^\omega(\Gamma^d)$ is called regular when its associated vector field χ is regular, and, furthermore:

(1) For all flowing edges γ, X has no singularities in $int(\gamma)$.

(2) X vanishes along every neutral edge.

In the sequel, we will restrict ourselves to regular vector fields $X \in \mathfrak{X}^\omega(\Gamma^d)$.

Now, given any corner $(v, \gamma, \sigma) \in C$, define:

$$\Sigma_{v,\gamma} = \left(\Psi^X_{v,\varepsilon}\right)^{-1}(\Pi_\gamma).$$

$\Sigma_{v,\gamma}$ is a cross section, transversal to the field X and it intersects γ at a single point, denoted by $q_{v,\gamma}$. Note that, as $v \in \gamma$, then, by **Remark 2.17**, $\Pi_\gamma \subset \Pi_v$. Recall that the function $\Psi^X_{v,\varepsilon}$ was defined in **Definition 2.13**. It has as target space the sector Π_v. So, the section $\Sigma_{v,\gamma} \subset N_v \setminus \partial\Gamma^d$ is well defined.

We use this notation:

Notation 6. Let us denote by $\varphi^t_X(x)$ the flow associated to the vector field $X \in \mathfrak{X}^\omega(\Gamma^d)$.

Now, take two vertexes $v, v' \in V$ and a flowing edge $v \xrightarrow{\gamma'} v'$. Let us denote by $D_{\gamma'}$ the points from $x \in int(\Sigma_{v,\gamma'})$ such that the forward orbit of X, that is, the set $\{\varphi^t_X(x)/\ t \geq 0\}$, has a first intersection, in a transversal way, with $\Sigma_{v',\gamma'}$. Then we define:

Definition 2.45. Let $\tau(x) = \min\{t > 0/\ \varphi^t_X(x) \in \Sigma_{v',\gamma'}\}$ be the first time for which the flow of X intersects $\Sigma_{v',\gamma'}$ transversally. Then, the partial Poincaré map $P_{\gamma'}$ is given by:

$$\begin{aligned} P_{\gamma'} : D_{\gamma'} \subset int(\Sigma_{v,\gamma'}) &\longrightarrow int(\Sigma_{v',\gamma'}) \\ x &\longmapsto P_{\gamma'}(x) = \varphi^{\tau(x)}_X(x). \end{aligned}$$

Analogously, given vertexes $v_0, v, v' \in V$ and flowing edges $v_0 \xrightarrow{\gamma} v$ and $v \xrightarrow{\gamma'} v'$, let $D_{\gamma,\gamma'}$ be the set of points $x \in int(\Sigma_{v,\gamma})$ such that the $\{\varphi_X^t(x)/\ t \geq 0\}$, has a first intersection, in a transversal way, with $\Sigma_{v,\gamma'}$. Define:

Definition 2.46. Let $\tau(x) = \min\{t > 0/\ \varphi_X^t(x) \in \Sigma_{v,\gamma'}\}$ be the first time for which the flow of X intersects $\Sigma_{v,\gamma'}$ transversally. Then, the partial Poincaré map $P_{\gamma,\gamma'}$ is given by:

$$\begin{aligned} P_{\gamma,\gamma'} : D_{\gamma,\gamma'} \subset int(\Sigma_{v,\gamma}) &\longrightarrow int(\Sigma_{v,\gamma'}) \\ x &\longmapsto P_{\gamma,\gamma'}(x) = \varphi_X^{\tau(x)}(x). \end{aligned}$$

Definition 2.47. Given a $\chi-$path $\xi = (\gamma_0, \gamma_1, \ldots, \gamma_m)$,the Poincaré map of X along ξ is defined by the composition

$$P_\xi = (P_{\gamma_m} \circ P_{\gamma_{m-1},\gamma_m}) \circ \ldots \circ (P_{\gamma_1} \circ P_{\gamma_0,\gamma_1}).$$

The domain of this map is denoted by D_ξ.

Remark 2.48. Given a structural set $S \subset E_\chi$ and two $S-$branches $\xi \neq \xi'$, then $D_\xi \cap D_{\xi'} = \emptyset$.

For a given path ξ, the asymptotic behaviour of the Poincaré map P_ξ along ξ is given by the corresponding Poincaré map π_ξ of the skeleton vector field χ. To make more precise what we understand for asymptotic behaviour, we introduce the following concept:

Definition 2.49. Consider a family of functions, depending on a parameter $\varepsilon > 0$, $F_\varepsilon : \mathcal{U}_\varepsilon \to \mathcal{U}_\varepsilon$. Let F be a function $F : \mathcal{U} \to \mathcal{U}$ and assume $\mathcal{U}_\varepsilon$ and $\mathcal{U}$ are linear spaces. We say that F_ε tends to F, as ε tends to zero, in the C^k topology, and we write $\lim\limits_{\varepsilon\to 0^+} F_\varepsilon = F$, when the following conditions are met:

(1) Domain convergence: for every $K \subset \mathcal{U}$ compact subset, we have that $K \subset \mathcal{U}_\varepsilon$, for every $\varepsilon > 0$ small enough.

(2) Uniform convergence on compact sets:

$$\lim_{\varepsilon\to 0^+} \max_{0\leq i\leq k} \sup_{u\in K} \left| D^i \left[F_\varepsilon(u) - F(u)\right]\right| = 0.$$

We say that the convergence holds in the C^∞ topology if it holds in the C^k topology, for all $k \geq 1$.

For an edge $\gamma \in E$, we define the sub-domain $\Pi_\gamma(\varepsilon)$ of Π_γ as

$$\Pi_\gamma(\varepsilon) = \{y \in \Pi_v / \ y_\sigma \geq \varepsilon, \text{ for all } \sigma \in F \text{ such that } \gamma \subset \sigma\}.$$

The next result, whose proof can be seen in [1], tells us that the asymptotic behaviour of the Poincaré map P_ξ is governed by π_ξ, as we mentioned earlier.

Proposition 2.50. *Given a $\chi-$path $\xi = (\gamma_0, \gamma_1, \ldots, \gamma_m)$ with $v_0 = s(\gamma_0)$ and $v_m = s(\gamma_m)$ and let $\mathcal{U}_\xi^\varepsilon$ be the domain of the map $F_\xi^\varepsilon : \Pi_{\gamma_0}(\varepsilon) \to \Pi_{\gamma_m}(\varepsilon)$, defined by the composition: $F_\xi^\varepsilon := \Psi_{v_m,\varepsilon}^X \circ P_\xi \circ (\Psi_{v_0,\varepsilon}^X)^{-1}$. Then we have the following limit in the C^k topology:*

$$\lim_{\varepsilon \to 0^+} (F_\xi^\varepsilon) = \pi_\xi.$$

Definition 2.51. Given $X \in \mathfrak{X}^\omega(\Gamma^d)$ a regular vector field with structural set $S \subset E_\chi$, we define the section $\Sigma_S = \cup_{\gamma \in S} \Sigma_\gamma$ and the domain $D_S = \cup_{\xi \in \mathscr{B}_S(\chi)} D_\xi$. The $S-$Poincaré map associated to X is defined as

$$\begin{aligned} P_S : D_S \subset \Sigma_S &\longrightarrow \Sigma_S \\ p &\longmapsto P_S(p) = P_\xi(p), \text{ when } x \in D_\xi. \end{aligned}$$

The following result can be obtained as a corollary of **Proposition 2.50** (see [1]).

Proposition 2.52. *Given $X \in \mathfrak{X}^\omega(\Gamma^d)$ a regular vector field with associated skeleton vector field χ and with structural set $S \subset E_\chi$, the following limit in the C^K topology holds:*

$$\lim_{\varepsilon \to 0^+} \Psi_\varepsilon^X \circ P_S \circ (\Psi_\varepsilon^X)^{-1} = \pi_S.$$

2.7 Projective Poincaré maps

Let $X \in \mathfrak{X}^\omega(\Gamma^d)$ be a regular vector field with skeleton χ and consider a $\chi-$structural set S. As before, $\mathbb{1} \in \mathbb{R}_+^F$ is the vector whose all coordinates are equal to 1. We write

$$\bar{u} := \mathbb{1}\ u = \sum_{\sigma \in F} u_\sigma.$$

Also, let $G = G_\chi = (V, E_\chi)$ denote the directed graph of χ and let $\mathscr{B}_S(\chi)$ be the set of all $S-$branches of G. Given a $\chi-$path $\xi = (\gamma_1, \ldots, \gamma_m)$ such that the cone Π_ξ has non-empty interior, define the $(d-2)$-simplex

$$\Delta_\xi^\chi = \{u \in int(\Pi_\xi) / \ \bar{u} = 1\}$$

and

$$\Delta_S^\chi = \bigcup_{\xi \in \mathscr{B}_S(\chi)} \Delta_\xi^\chi.$$

In an analogous way, define, for each edge $\gamma \in E_\chi$

$$\Delta_\gamma = \{u \in int(\Pi_\gamma) / \ \bar{u} = 1\}$$

and set

$$\Delta_S = \bigcup_{\gamma \in S} \Delta_\gamma.$$

Definition 2.53. For a $\chi-$path $\xi, = (\gamma_1, \ldots, \gamma_m)$, the projective Poincaré map along ξ is defined as the map

$$\begin{aligned} \hat{\pi}_\xi : \Delta_\xi^\chi \subset \Delta_{\gamma_1} &\longrightarrow \Delta_{\gamma_m} \\ u &\longmapsto \hat{\pi}_\xi(u) = \frac{\pi_\xi(u)}{\overline{\pi_\xi(u)}} \end{aligned}$$

The projective $S-$Poincaré map is the map

$$\begin{aligned} \hat{\pi}_S : \Delta_S^\chi \subset \Delta_S &\longrightarrow \Delta_S \\ u &\longmapsto \hat{\pi}_S(u) = \hat{\pi}_\xi(u), \text{ for all } u \in \Delta_\xi^\chi. \end{aligned}$$

Chapter 3

Dynamics of human decisions

This chapter is organized as follows. In the first two sections, we review the decision model from [19], which is a 2×2 dimensional model. We briefly recall the results appearing in the mentioned source according to pure and mixed strategies.

In the third section, we generalize this decision model to the case with n strategies and k different individuals. We point out that the generalized model can be written as a polymatrix equation and thus, the work presented in chapter 2 can be used to study the asymptotic behaviour of the general $k \times n$ model. In the fourth section we try to unify the notation from this model and from the polymatrix equation.

In the last section, we restrict to the model with dimension $k \times 2$ and we extend the definitions of the Nash domains from the 2×2 case. Further, we give necessary and sufficient conditions for the existence of pure Nash equilibria in all point for the case $k = 2$. As a future work, we can extend this to $k > 2$.

3.1 The Yes-No decision model

Let us consider a population I consisting of two types of individuals $T = \{t_1, t_2\}$. Let $I_1 = \{1, ..., n_1\}$ be the number of individuals with type t_1 and $I_2 = \{1, ..., n_2\}$ the number of individuals with type t_2. Then we have $I = I_1 \sqcup I_2$. Each individual $i \in I$ has to make one decision $d \in D = \{Y, N\}$[1]

[1] Alternative models consider an only individual with type t_p which has to make n_p decisions.

Before continuing, let us introduce some constants, which indicate the preferences of an individual $i \in I$ making a decision $d \in D$.

Notation 7. γ_p^d indicates how much a type t_p individual likes or dislikes to make a decision d.

β_{pq}^d indicates how much a type t_p individual, whose decision is d, likes or dislikes that a type t_q individual makes the same decision.

$\overline{\beta}_{pq}^d$ indicates how much a type t_p individual, whose decision is d, likes or dislikes that a type t_q individual makes the other decision.

Remark 3.1. The values γ_p^d tell us the preferences of the individuals, that is, the taste type of the individuals. On the other hand, the values β_{pq}^d and $\overline{\beta}_{pq}^d$ tell us with whom the individuals want to share their decisions, that is, the crowding type of the individuals.

We are going to describe the pure decisions of the individuals in terms of a strategic map $S : I \to D$, which associates to each individual i its decision $S(i) = d$.

Let us denote by $\mathbf{S}$ the space of all possible strategies S.

Definition 3.2. Given a strategy $S \in \mathbf{S}$, we call strategic decision vector associated to S to the vector $(l_1, l_2) = (l_1(S), l_2(S))$, where l_i is the number of individuals with type t_1 who make decision Y.

The set of all possible strategic decision vectors is

$$\mathbf{O} = \{0, 1, \ldots, n_1\} \times \{0, 1, \ldots, n_2\}$$

Notation 8. If $d \in \{Y, N\}$ and $i, j \in \{1, 2\}$:

$\omega_1^Y = \gamma_1^Y + \overline{\beta}_{11}^Y (n_1 - 1) + \overline{\beta}_{12}^Y n_2,$

$\omega_1^N = \gamma_1^N + \overline{\beta}_{11}^N (n_1 - 1) + \overline{\beta}_{12}^N n_2,$

$\omega_2^Y = \gamma_2^Y + \overline{\beta}_{22}^Y (n_2 - 1) + \overline{\beta}_{21}^Y n_1,$

$\omega_2^N = \gamma_2^N + \overline{\beta}_{22}^N (n_2 - 1) + \overline{\beta}_{21}^N n_1,$

$\alpha_{ij}^d = \beta_{ij}^d - \overline{\beta}_{ij}^d.$

The utility function of the individuals are given by the following expressions: If the individual has type t_1, then its utility is given by the function:

$$\begin{aligned}
U_1 : D \times O &\longrightarrow \mathbb{R} \\
(Y;(l_1,l_2)) &\longmapsto \gamma_1^Y + \beta_{11}^Y(l_1-1) + \beta_{12}^Y l_2 + \overline{\beta}_{11}^Y(n_1-l_1) + \overline{\beta}_{12}^Y(n_2-l_2) \\
&= \omega_1^Y + \alpha_{11}^Y(l_1-1) + \alpha_{12}^Y l_2 \\
(N;(l_1,l_2)) &\longmapsto \gamma_1^N + \beta_{11}^N(n_1-l_1-1) + \beta_{12}^N(n_2-l_2) + \overline{\beta}_{11}^N l_1 + \overline{\beta}_{12}^N l_2 \\
&= \omega_1^N + \alpha_{11}^N(n_1-l_1-1) + \alpha_{12}^Y(n_2-l_2).
\end{aligned}$$

while the utility of the t_2 individuals is given by:

$$\begin{aligned}
U_2 : D \times O &\longrightarrow \mathbb{R} \\
(Y;(l_1,l_2)) &\longmapsto \gamma_2^Y + \beta_{22}^Y(l_2-1) + \beta_{21}^Y l_1 + \overline{\beta}_{22}^Y(n_2-l_2) + \overline{\beta}_{21}^Y(n_1-l_1) \\
&= \omega_2^Y + \alpha_{22}^Y(l_2-1) + \alpha_{21}^Y l_1 \\
(N;(l_1,l_2)) &\longmapsto \gamma_2^N + \beta_{22}^N(n_2-l_2-1) + \beta_{21}^N(n_1-l_1) + \overline{\beta}_{22}^N l_2 + \overline{\beta}_{21}^N l_1 \\
&= \omega_2^N + \alpha_{22}^N(n_2-l_2-1) + \alpha_{21}^Y(n_1-l_1).
\end{aligned}$$

Thus, given a strategy $S \in \mathbf{S}$, the utility of the individual $i \in I$, with type $t_{p(i)}$, using the strategy S, is: $U_{p(i)}(S(i); l_1(S), l_2(S))$, where $p(i) \in \{1, 2\}$ means the type of individual.

Definition 3.3. We set $x = \omega_1^Y - \omega_1^N$ to be the horizontal relative decision preference of the individuals with type t_1 and $y = \omega_2^Y - \omega_2^N$ to be the vertical relative decision preference of the individuals with type t_2. Also, set $A_{ij} = \alpha_{ij}^Y + \alpha_{ij}^N, i, j \in \{1, 2\}$ to be the coordinates of the influence matrix.

We observe that the sign of x and y determine the preference of the individuals to take one or another decision:

If $x > 0$, then individuals with t_1 prefer to choose Y without taking into account the influence of the others. If $x = 0$, the individuals with type t_1 are indifferent to decide Y or N without taking into account the influence of the others. If $x < 0$, individuals with type t_1 prefer to decide N, without taking into account the influence of the others.

On the other hand, A_{ij} determines the influence of some individuals over others: If $A_{ij} > 0$, the individuals with type t_j have a positive influence over the utility of the individuals with type t_i. If $A_{ij} = 0$, the individuals with type t_j do not affect the utility of the individuals with type t_i. If $A_{ij} < 0$, the individuals with type t_j have a negative influence over the utility of the individuals with type t_i.

3.1.1 Pure strategies

Now we will introduce here the concept of Nash equilibria. Two types of equilibria are considered: cohesive and disparate. We say that one strategy is cohesive when individuals of the same type always prefer to make the same decisions. Otherwise, the strategy is called disparate.

Nash equilibria for pure strategies is defined as follows:

Definition 3.4. A strategy $S^* : I \to D$ is a pure Nash equilibria if for all individual $i \in I$ and all strategy $S \in \mathbf{S}$, such that $S^*(j) = S(j)$ for $j \in I \setminus \{i\}$, we have that

$$U_i(S^*) \geq U_i(S).$$

That is, if only one individual change its strategy from a Nash equilibria, then he obtains less payoff.

The following concept will also be important for our purpose:

Definition 3.5. We call Nash domain of a strategy $S \in \mathbf{S}$ to the set:

$$N(S) = \{(x, y) \in \mathbb{R}^2 / S \text{ is a Nash equilibria}\}.$$

Cohesive strategies

We observe that only four cohesive strategies are possible:

(Y, Y) strategy, which happens when all individuals make the decision Y.

(Y, N) strategy, if all individuals with type t_1 make the decision Y and all individuals with type t_2 make the decision N.

(N, Y) strategy, when all individuals with type t_1 make the decision N and all individuals with type t_2 make the decision Y

(N, N) strategy, if all individuals make the decision N.

We will start with the (Y, Y) strategy :

Proposition 3.6. *Given the strategy (Y, Y), its Nash domain is given by:*

$$N(Y, Y) = \{(x, y) \in \mathbb{R}^2 / x \geq H(Y, Y), y \geq V(Y, Y)\},$$

where $H(Y, Y)$ and $V(Y, Y)$ are, respectively, the horizontal and vertical strategic thresholds for (Y, Y) and are given by:

$$H(Y, Y) = -\alpha_{11}^{Y}(n_1 - 1) - \alpha_{12}^{Y} n_2$$
$$V(Y, Y) = -\alpha_{22}^{Y}(n_2 - 1) - \alpha_{21}^{Y} n_1$$

Proof. It follows directly from the Nash equilibrium definition. $\square$

Similar results are obtained for the rest of the cohesive strategies, as it is shown on [19].

Disparate strategies

An (l_1, l_2) strategic set is the set of all pure strategies $S \in \mathbf{S}$ with $l_1(S) = l_1$ and $l_2(S) = l_2$. An (l_1, l_2) cohesive strategic set is an (l_1, l_2) strategic set with $l_1 \in \{0, n_1\}$ and $l_2 \in \{0, n_2\}$. An (l_1, l_2) disparate strategic set is an (l_1, l_2) strategic set that is not cohesive. We observe that a cohesive strategic set has a single strategy and a disparate strategic set has more than one strategy. Since individuals with the same type are identical, a strategy to be a Nash equilibrium depends only of the number of individuals of each type that decide either Y or N, and not of the individual who is making the decision.

Definition 3.7. An (l_1, l_2) pure Nash equilibrium set is an (l_1, l_2) strategic set whose strategies are Nash equilibria. The pure Nash domain $N(l_1, l_2)$ is the set of all pairs (x, y) for which the (l_1, l_2) strategic set is a Nash equilibrium set.

The (l_1, l_2) pure Nash equilibrium set is cohesive if $l_1 \in \{0, n_1\}$ and $l_2 \in \{0, n_2\}$. The (l_1, l_2) pure Nash equilibrium set is disparate if $l_1 \notin \{0, n_1\}$ or $l_2 \notin \{0, n_2\}$.

Now we state two results whose proof follows directly from Nash equilibrium definition and that can be found in [19].

Lemma 3.8. *Let (l_1, l_2) be a Nash equilibrium.*

(i) If $A_{11} > 0$, then $l_1 \in \{0, n_1\}$.

(ii) If $A_{22} > 0$, then $l_2 \in \{0, n_2\}$.

Furthermore, if $A_{11} > 0$ and $A_{22} > 0$, then (l_1, l_2) is cohesive.

Thus, if $A_{11} > 0$ and $A_{22} > 0$, then there are not disparate Nash equilibria.

Let $C \in \mathbb{R}^2$ be given by $C = (H(N, N), V(N, N))$. The disparate vector $\vec{Z}(l_1, l_2)$ is defined by

$$\vec{Z}(l_1, l_2) = -l_1(A_{11}, A_{21}) - l_2(A_{12}, A_{22}).$$

Lemma 3.9. *Let $l_1 \in \{0, \ldots, n_1 - 1\}$ and $l_2 \in \{0, \ldots, n_2 - 1\}$.*

(i) If $A_{11} \le 0$, then the disparate Nash domain $N(l_1, 0)$ is given by

$$N(l_1, 0) = \{C + \vec{Z}(l_1, 0) + (pA_{11}, q) : p \in [0, 1]; q \in (-\infty, 0]\}.$$

and the disparate Nash domain $N(l_1, n_2)$ is given by

$$N(l_1, n_2) = \{C + \vec{Z}(l_1, n_2) + (pA_{11}, q) : p \in [0, 1]; q \in (0, +\infty]\}.$$

(ii) If $A_{22} \le 0$, then the disparate Nash domain $N(0, l_2)$ is given by

$$N(0, l_2) = \{C + \vec{Z}(0, l_2) + (p, qA_{22}) : q \in [0, 1]; p \in (-\infty, 0]\}.$$

and the disparate Nash domain $N(n_1, l_2)$ is given by

$$N(n_1, l_2) = \{C + \vec{Z}(n_1, l_2) + (p, qA_{22}) : q \in [0, 1]; p \in (0, +\infty]\}.$$

(iii) If $A_{11} \le 0$ and $A_{22} \le 0$, then the disparate Nash domain $N(l_1, l_2)$ is given by:

$$N(l_1, l_2) = \{C + \vec{Z}(l_1, l_2) + (pA_{11}, qA_{22}) : p, q \in [0, 1]\}.$$

Thus, if $A_{11} \le 0$ and $A_{22} \le 0$, then for every (l_1, l_2) disparate strategic set there are relative preferences for which (l_1, l_2) is a Nash equilibrium set.

3.1.2 Mixed strategies

Now we will introduce mixed strategies and mixed Nash equilibria. The idea which underlies here is to assign some probabilities to the decision making. Mixed strategies are given by a function $S : I \to [0, 1]$ which associates to each individual $i \in I_1$ its probability to decide Y, which is denoted by p_i and to each individual in I_2 its probability to decide Y, denoted by q_j. Thus, each individual $i \in I$ decides N with probability $1 - p_i$ and each individual $j \in I_2$ decides N with probability $1 - q_j$. It is assumed that the decisions are made independently.

We will introduce here some more notation:

Notation 9. We denote:

$P = \sum_{i=1}^{n_1} p_i$ and $P_i = P - p_i$.

$Q = \sum_{j=1}^{n_2} q_j$ and $Q_i = Q - q_j$.

Now we are going to define an analogous concept to the utility function for the pure strategies case. This is the fitness function:

Definition 3.10. For every individual $i \in I_1$, the Y-fitness function $f_{Y,1}$ is given by:

$$\begin{aligned} f_{Y,1} : [0,1] \times [0, n_1] \times [0, n_2] &\longrightarrow \mathbb{R} \\ (p_i; P, Q) &\longmapsto \omega_1^Y + \alpha_{11}^Y P_i + \alpha_{12}^Y Q \end{aligned}$$

and the N-fitness function $f_{N,1}$ is given by:

$$\begin{aligned} f_{N,1} : [0,1] \times [0, n_1] \times [0, n_2] &\longrightarrow \mathbb{R} \\ (p_i; P, Q) &\longmapsto \omega_1^N + \alpha_{11}^N (n_1 - 1 - P_i) + \alpha_{12}^N (n_2 - Q). \end{aligned}$$

For every individual $j \in I_2$, the Y-fitness function $f_{Y,2}$ is given by:

$$\begin{aligned} f_{Y,2} : [0,1] \times [0, n_1] \times [0, n_2] &\longrightarrow \mathbb{R} \\ (q_j; P, Q) &\longmapsto \omega_2^Y + \alpha_{22}^Y Q_j + \alpha_{21}^Y P \end{aligned}$$

and the N-fitness function $f_{N,2}$ is given by:

$$f_{N,2} : [0,1] \times [0,n_1] \times [0,n_2] \longrightarrow \mathbb{R}$$
$$(q_j; P, Q) \longmapsto \omega_2^N + \alpha_{22}^N(n_2 - 1 - Q_j) + \alpha_{21}^N(n_1 - P).$$

Now we are going to introduce one concept that in [19] is referred as utility function, but some authors call it average payoff or average utility.

We have the following result, whose proof can be seen in [19] and which gives us one formula to compute more easily the utility functions:

Lemma 3.11. *Let $S : I \to [0,1]$ be a mixed strategy. For every individual $i \in I_1$, its utility function is given by:*

$$U_1 : [0,1] \times [0,n_1] \times [0,n_2] \longrightarrow \mathbb{R}$$
$$(p_i; P, Q) \longmapsto p_i\, f_{Y,1}(p_i; P, Q) + (1 - p_i)\, f_{N,1}(p_i; P, Q),$$

while for every individual $j \in I_2$, its utility function is given by:

$$U_2 : [0,1] \times [0,n_1] \times [0,n_2] \longrightarrow \mathbb{R}$$
$$(q_j; P, Q) \longmapsto q_j\, f_{Y,2}(q_j; P, Q) + (1 - q_j)\, f_{N,2}(q_j; P, Q),$$

Definition 3.12. A strategy $S^* : I \to [0,1]$ is a mixed Nash equilibria if for all individual $i \in I$ and all strategy $S \in \mathbf{S}$, such that $S^*(j) = S(j)$ for $j \in I \setminus \{i\}$, we have that

$$U_i(S^*) \geq U_i(S).$$

That is, if only one individual change its strategy from a Nash equilibria, then he obtains less payoff.

3.1.3 Strategic sets

Definition 3.13. We call (l_1, l_2)-strategic pure set to the set of all strategies $S \in \mathbf{S}$ such that $l_1(S) = l_1$ and $l_2(S) = l_2$, i.e., the strategies for which there are l_1 individuals of type t_1 and l_2 individuals of type t_2 choosing Y.

Definition 3.14. We call $(l_1, k_1, p; l_2, k_2, q)$ mixed strategy set to the set of all strategies which satisfy:

i) $l_1 = \#\{i \in I_1/p_i = 1\}$ and $k_1 = \#\{i \in I_1/p_i = p\}$.

ii) $l_2 = \#\{j \in I_2/q_j = 1\}$ and $k_2 = \#\{j \in I_2/q_j = q\}$.

iii) $n_1 - (l_1 + k_1) = \#\{i \in I_1/p_i = 0\}$ and $n_2 - (l_2 + k_2) = \#\{j \in I_2/q_j = 0\}$.

Remark 3.15. The following is derived from the two previous definitions:

1. The $(l_1, k_1, p; l_2, k_2, q)$ mixed strategy set consists of strategies for which there are l_1 individuals with type t_1 and l_2 individuals with type t_2 choosing Y, $n_1 - (l_1 + k_1)$ individuals with type t_1 and $n_2 - (l_2 + k_2)$ individuals with type t_2 choosing N and k_1 individuals with type t_1 and k_2 individuals with type t_2 choosing Y with probabilities p and q respectively.

2. If $p, q \in \{0, 1\}$, then the $(l_1, k_1, p; l_2, k_2, q)$ mixed strategy set coincides with the $(l_1 + pk_1, l_2 + qk_2)$ pure strategic set.

3. From 2. we deduce that, if $k_1 = k_2 = 0$, then the strategic $(l_1, 0, p; l_2, 0, q)$ set coincides with the pure strategic set (l_1, l_2).

We note that, because individuals from the same group are indistinguishable, if a mixed strategy is contained in the $(l_1, k_1, p; l_2, k_2, q)$ mixed strategy set is a Nash equilibrium, then all the strategies in that strategic set are also Nash equilibria.

We introduce now more notation, motivated by the previous remark, and which will be useful in the following.

Notation 10. Given one strategy from the $(l_1, k_1, p; l_2, k_2, q)$ mixed strategy set, we define:

$$v[1l] = (v_1^{1l}, \ldots, v_{l_1}^{1l}), \quad v[1m] = (v_1^{1m}, \ldots, v_{n_1-(l_1+k_1)}^{1m}), \quad v[1r] = (v_1^{1r}, \ldots, v_{k_1}^{1r})$$

$$v[2l] = (v_1^{2l}, \ldots, v_{l_2}^{2l}), \quad v[2m] = (v_1^{2m}, \ldots, v_{n_2-(l_2+k_1)}^{2m}), \quad v[2r] = (v_1^{2r}, \ldots, v_{k_2}^{2r})$$

We define the vectors $v[1] \in \mathbb{R}_1^n$, $v[2] \in \mathbb{R}_2^n$ and $v \in \mathbb{R}^{n_1+n_2}$ by:

$$v[1] = (v[1l], v[1m], v[1r]), \;\; v[2] = (v[2l], v[2m], v[2r]) \text{ and } v = (v[1], v[2]).$$

Let us now define $V[1]$ and $V[2]$ by:

$$\begin{aligned} V[1] &= \sum_{i=1}^{l_1} v_i^{1l} + \sum_{j=1}^{n_1-(l_1+k_1)} v_j^{1m} + \sum_{k=1}^{k_1} v_k^{1r}, \\ V[2] &= \sum_{i=1}^{l_2} v_i^{2l} + \sum_{j=1}^{n_2-(l_2+k_2)} v_j^{2m} + \sum_{k=1}^{k_2} v_k^{2r}. \end{aligned}$$

Remark 3.16. The vectors $v[1l], v[2l]$ are formed by the probabilities of individuals with type t_1 and t_2, respectively, that are choosing Y.

The vectors $v[1m], v[2m]$ are formed by the probabilities of individuals with type t_1 and t_2, respectively, that are choosing N.

The vectors $v[1r], v[2r]$ are formed by the probabilities of individuals with type t_1 and t_2 that are choosing Y with probabilities p and q, respectively.

Always with the previous remark in mind, it is clear that $v[1]$ corresponds to the vector $(p_1, \ldots, p_{n_1})$ after applying (possibly) a permutation to its components. The same holds for $v[2]$ and the vector $(q_1, \ldots, q_{n_2})$.

Finally, $V[1]$ and $V[2]$ correspond to P and Q, respectively.

Remark 3.17. When $k_1 = k_2 = 0$, we have that $v[1r] = \emptyset$ and $v[2r] = \emptyset$, so $v[1] = (v[1l], v[1m])$ and $v[2] = (v[2l], v[2m])$.

Now, we give the following definition:

Definition 3.18. The $(l_1, k_1, p; l_2, k_2, q)$ canonical strategy is any strategy from the $(l_1, k_1, p; l_2, k_2, q)$ mixed strategy set, that is, one strategy which satisfies the following conditions:

- for all $i \in \{1, \ldots, l_1\}, j \in \{1, \ldots, n_1 - (l_1 + k_1)\}$ and $k \in \{1, \ldots, k_1\}$,

$$v_i^{1l} = 1, \; v_j^{1m} = 0 \text{ and } v_k^{1r} = p.$$

- for all $i \in \{1, \ldots, l_2\}, j \in \{1, \ldots, n_2 - (l_2 + k_2)\}$ and $k \in \{1, \ldots, k_2\}$,

$$v_i^{2l} = 1, \; v_j^{2m} = 0 \text{ and } v_k^{2r} = q.$$

3.2 The dynamics of the decisions

We will assume now that the frequencies of choosing Y, that is, the values p_i and q_j, are changing with the time. Thus, the individuals can now change their decisions with the time. This leads to dynamics in the decision making:

$$\begin{cases} \dot{p}_i = p_i\big(f_{Y,1}(p_i;P,Q) - U_1(p_i;P,Q)\big) \\ \\ \dot{q}_j = q_j\big(f_{Y,2}(q_j;P,Q) - U_2(q_j;P,Q)\big), \end{cases}$$

which can be rewritten as:

$$\begin{cases} \dot{p}_i = p_i(1-p_i)\big(f_{Y,1}(p_i;P,Q) - f_{N,1}(p_i;P,Q)\big) \\ \\ \dot{q}_j = q_j(1-q_j)\big(f_{Y,2}(q_j;P,Q) - f_{N,2}(q_j;P,Q)\big), \end{cases}$$

using the expressions for the utility functions. Now, using the expressions for the fitness functions, we can write the dynamics as the following O.D.E.s system:

$$\begin{cases} \dot{p}_i = p_i(1-p_i)\big(P_iA_{11} + QA_{12} + x - H(N,N)\big) \\ \\ \dot{q}_j = q_j(1-q_j)\big(Q_jA_{22} + PA_{21} + y - V(N,N)\big), \end{cases} \tag{3.1}$$

where $i \in \{1,\dots,n_1\}$ and $j \in \{1,\dots,n_2\}$.

We will denote this dynamics as $\dot{S} = G(S;x,y)$, to emphasize the fact that the strategies are changing with the time. Here, $G : [0,1]^{n_1+n_2} \times \mathbb{R} \times \mathbb{R} \to [0,1]^{n_1+n_2}$ is the vector field associated with (3.1).

Remark 3.19. We note that, as $p_i, q_j \in [0,1]$, for $i \in \{1,\dots,n_1\}$ and $j \in \{1,\dots,n_2\}$, then the system (3.1) has as phase space the n_1+n_2 dimensional cube $[0,1]^{n_1+n_2}$. Also, we note that the faces of this cube (ie, points where at least one p_i or q_j vanish) and, consequently, the cube itself are invariant by the flow generated by the system. This suggests us that the work explained in chapter 2 can be applied to the system (3.1).

Definition 3.20. We say that a strategy $S : I \to [0,1]$ is a dynamical equilibrium of (3.1) if $G(S; x, y) = 0$, that is:

$$\begin{cases} \dot{p_i} = p_i(1-p_i)\big(P_i A_{11} + Q A_{12} + x - H(N,N)\big) = 0 \\ \\ \dot{q_j} = q_j(1-q_j)\big(Q_j A_{22} + P A_{21} + y - V(N,N)\big) = 0, \end{cases}$$

for $i \in \{1, \dots, n_1\}$ and $j \in \{1, \dots, n_2\}$.

We can also consider the linearised system of (3.1), denoted by $\dot{S} = DG(S; x, y)$, where DG denotes the Jacobian matrix of G.

Definition 3.21. We say that an equilibrium strategy S is strongly stable if all the eigenvalues of $DG(S; x, y)$ have negative real parts. S is called strongly unstable if $DG(S; x, y)$ has at least one eigenvalue with positive real part.

Definition 3.22. We call equilibria domain $E(l_1 k_1, p; l_2, k_2, q)$ to the set of all pairs $(x, y) \in \mathbb{R}^2$ for which the strategies contained in the $(l_1 k_1, p; l_2, k_2, q)$ strategic set are equilibria of the dynamics.

Similarly, the strongly stable domain $S(l_1 k_1, p; l_2, k_2, q)$ is the set of all pairs $(x, y) \in \mathbb{R}^2$ for which the strategies contained in the $(l_1 k_1, p; l_2, k_2, q)$ strategic set are strongly stable equilibria of the dynamics.

The strongly unstable domain $U(l_1 k_1, p; l_2, k_2, q)$ is the set of all pairs $(x, y) \in \mathbb{R}^2$ for which the strategies contained in the $(l_1 k_1, p; l_2, k_2, q)$ strategic set are strongly unstable equilibria of the dynamics.

Remark 3.23. We have the following chain of inclusions:

$$S(l_1, k_1, p; l_2, k_2, q) \subset N(l_1, k_1, p; l_2, k_2, q) \subset E(l_1, k_1, p; l_2, k_2, q)$$

Using the **Notation 10**, the dynamics (3.1) can be written as:

$$\begin{cases} \dot{v}_{i_1}^{1s} = v_{i_1}^{1s}(1 - v_{i_1}^{1s})\big((V[1] - v_{i_1}^{1s})A_{11} + V[2]A_{12} + x - H(N,N)\big),\ i_1 \in I_1 \\ \\ \dot{v}_{i_2}^{2s} = v_{i_2}^{2s}(1 - v_{i_2}^{2s})\big((V[2] - v_{i_2}^{2s})A_{22} + V[2]A_{21} + y - V(N,N)\big),\ i_2 \in I_2, \end{cases} \tag{3.2}$$

where $s \in \{l, m, r\}$ and $(x, y) \in \mathbb{R}^2$.

3.3 General decision model

Now we consider a more general scenario, consisting of the same game as explained in the previous sections, with the difference that now, we have a number n_I of individuals, each of which can decide between a number n_D of decisions. The starting point is the model introduced in [20]. We give the equations of the evolution of strategy frequencies for that game and we give an alternative version for it. This is new with respect to the mentioned source.

Two variants are considered in this work. The difference between them will lay on the called Independence of Irrelevant Alternatives (IIA) condition. It consists of the restriction of social interactions to those individuals who make the same decisions. IIA condition is satisfied in the model presented in [20].

Let us denote by $I = \{n_1, \dots, n_I\}$ the individuals of the population, who are making one decision between the possible decisions $D = \{1, \dots, n_D\}$. Associated to the individuals, there is a strategy map, which associates to each individual the decision she is making:

$$\begin{aligned} S : I &\longrightarrow D \\ i &\longmapsto S(i) \equiv S_i \end{aligned}$$

We will note by $S^{-1}(S_i) \subset I$ the individuals that make a decision S_i.

Version with IIA condition The first variant considers that the IIA condition is satisfied. We are here inspired in [20].

For this situation, the utility function for every individual results:

$$\begin{aligned} U : I \times D &\longrightarrow \mathbb{R} \\ (i,d) &\longmapsto u(i,d) = \omega_i^d + \sum_{j \in S^{-1}(s_i) \setminus \{i\}} \alpha_{ij}^d, \end{aligned}$$

where ω_i^d represents how much an individual i likes to make decision d and α_{ij}^d indicates how much an individual i likes that an individual j share with her a decision d. By p_i^d we will represent the probability of an individual i to choose the strategy d.

We denote by

$$u_p(i,d) = \omega_i^d + \sum_{j \in S^{-1}(s_i)\setminus\{i\}} \alpha_{ij}^d p_j^d$$

the gain for an individual who is making a pure decision d. The average utility will be denoted by

$$\bar{u}(i) = \sum_{d \in D} p_i^d u_p(i,d).$$

In this fashion, we arrive to the usual replicator equation:

$$\frac{\dot{p}_i^d}{p_i^d} = u_p(i,d) - \bar{u}(i).$$

Version without the IIA condition and with types In the second case we introduce here, we assume that IIA condition is not satisfied. We use the same notation as in the other case, but know, we introduce groups of individuals. Namely, the individual set can be decomposed in n groups of individuals of type t_i: $I = I_{t_1} \cup \ldots \cup I_{t_k}$. We denote by T the set of the possible groups, $T = \{t_1, \ldots, t_k\}$.

The utility function here results:

$$\begin{aligned} U : I \times D &\longrightarrow \mathbb{R} \\ (i,d) &\longmapsto u(i,d) = \omega_t^d + \sum_{d' \in D} \sum_{\substack{t' \in T \\ t' \neq t}} \alpha_{tt'}^{dd'} l_{t'}^{d'}, \text{ whenever } i \in I_t, \end{aligned}$$

where $l_{t'}^{d'}$ is the number of individuals from t' choosing $d'$and $\alpha_{tt'}^{dd'}$ denotes the influence over an individual $i \in I_t$ who makes decision d of another individual $j \in I_{t'}$ making decision d'. The fact that we do not consider the case $t' = t$ is in relation with the hypothesis we made: the individuals of a certain type do not have effect over the utility of individuals of the same type.

Introducing the notation:

$$P_{t'}^{d'} = \sum_{i \in I_{t'}} p_i^{d'},$$

as before, we have that

$$u_p(i,d) = \omega_t^d + \sum_{d' \in D} \sum_{\substack{t' \in T \\ t' \neq t}} \alpha_{tt'}^{dd'} P_{t'}^{d'}, \text{ whenever } i \in I_t,$$

represents the gain for an individual who is making a pure decision d and

$$\bar{u}(i) = \sum_{d \in D} p_i^d u_p(i,d)$$

represents the average gain for that individual. The replicator equation we purpose in this case is

$$\frac{\dot{p}_i^d}{p_i^d} = u_p(i,d) - \bar{u}(i).$$

Remark 3.24. This general model is contained in the polymatrix replicator model, as $\omega_i^d = \sum_{d' \in D} \omega_i^d p_i^{d'}$, as $\sum_{d' \in D} p_i^{d'} = 1$. Then, the utility can be written as

$$u(i,d) = \sum_{d' \in D} \omega_i^d p_i^{d'} + \sum_{d' \in D} \sum_{\substack{j=1 \\ j \neq i}}^{k} \alpha_{ij}^{dd'} p_j^{d'},$$

just as in the polymatrix equation.

Remark 3.25. Observe that the two following facts are satisfied:

- $\sum_{d \in D} p_i^d = 1.$
- $\sum_{d \in D} \dot{p}_i^d = 0.$

3.4 The general model as a polymatrix replicator

Our aim here is to determine whether the general $k \times n$ model (without IIA condition), that is, with K different types of individuals deciding between a set of n decisions, presented in the last section is equivalent to the polymatrix replicator that we previously introduced. Observing carefully both models, we arrive to the conclusion that, the general model will be equivalent to the polymatrix replicator as soon as we make some restrictions.

First of all, it seems reasonable to consider that the groups I_t, with $t \in T$ are formed only by one individual. In this situation, we have a bijection between individuals and groups, that is, if $i \in I_t$, then we can relate $i \leftrightarrow t$.

The **Remark 3.24** is crucial in this discussion, as it implies that the term $\omega_t^d = 0$ does not represent a problem, despite the polymatrix replicator does not have this term.

The main difficulty we face here is the different notations adopted in both models. We tried to make an effort in unifying this two different notations presenting a new one.

Notation 11. Let us denote:

- $\vec{Q}_i = \left(\frac{\dot{p}_i^{d_1}}{p_i^{d_1}}, \ldots, \frac{\dot{p}_i^{d_n}}{p_i^{d_n}}\right)^T$ and $\vec{Q} = \left(\vec{Q}_1, \ldots, \vec{Q}_k\right)^T$.
- $\vec{P}_i = \left(p_i^{d_1}, \ldots, p_i^{d_n}\right)^T$ and $\vec{P} = \left(\vec{P}_1, \ldots, \vec{P}_k\right)^T$.
- $[\vec{P}_i^T] = \left(\vec{P}_i^T, \ldots, \vec{P}_i^T\right)^T$ and $\vec{Q}_i^{d_l} = \frac{\dot{p}_i^{d_l}}{p_i^{d_l}}$.

Let us now introduce the payoff matrix A, to make clear the equivalence with the polymatrix replicator.

Let us notate

$$A^{ij} = \begin{pmatrix} \alpha_{ij}^{d_1 d_1} & \ldots & \alpha_{ij}^{d_1 d_n} \\ \vdots & & \vdots \\ \alpha_{ij}^{d_n d_1} & \ldots & \alpha_{ij}^{d_n d_n} \end{pmatrix} = (\alpha_{ij}^{d_l d_{l'}})_{l,l' \in \{1,\ldots,n\}},$$

$$A^i = \left(A^{i1} | \ldots | A^{ik}\right)$$

and

$$A = (A^1 | \ldots | A^k)^T = \begin{pmatrix} A^{11} & \ldots & A^{1k} \\ \vdots & & \vdots \\ A^{i1} & \ldots & A^{ik} \\ \vdots & & \vdots \\ A^{k1} & \ldots & A^{kk} \end{pmatrix}$$

We have in our case the restriction $A^{ii} = 0$, as groups are formed by individuals and thus, no possible interaction between members from the same group can occur.

With this notations, we can write the general model as:

$$\vec{Q}_i^{d_l} = (A^i \vec{P})_{d_l} + [\vec{P}_i^T](A^i \vec{P}).$$

Remark 3.26. We observe that $A_{ij} = Tr(A^{ij}) = \alpha_{ij}^{d_1 d_1} + \ldots + \alpha_{ij}^{d_n d_n}$.

3.5 Pure Nash equilibria

Let us consider here the last equations on the case $k \times 2$, which corresponds to the situation where there are k individuals who can choose between two different decisions, say Y and N. We assume through this section that we are under the IIA condition. In our equation, this means that $\alpha_{ij}^{dd'} = 0$, for all $d' \neq d$ and all i, j. We restrict ourselves to the case with two decisions, as we already have a knowledge of the situation of Nash equilibria domains, as appears in [19]. Though the n decisions case can be studied, it would require to think everything from the beginning.

Let $(\vec{P})_i$ be the coordinate i of $\vec{P}$.

Notation 12. Let

- $\vec{e}_i = (|A_{1i}|, \ldots, |A_{ki}|)$
- $\vec{c}_i = -(0, \ldots, 0, |A_{ii}|, 0, \ldots, 0) = -|A_{ii}|\hat{c}_i$, where $\hat{c}_i$ are the vectors from the canonical basis of $\mathbb{R}^2$.
- $\vec{E}_i = (|A_{1i}|, \ldots, 0, \ldots, |A_{ki}|) = \vec{e}_i + \vec{c}_i$.
- $H_i = \{(x_1, \ldots, x_k) \in \mathbb{R}^k / \ x_i = 0\}$.
- $H_i(\vec{P}) = \{(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_k) / \ x_i = (\vec{P})_i\}$, that is, the hyperplane parallel to H_i passing through $\vec{P}$.

Remark 3.27. Observe that $\vec{E}_j \in H_j$ and $\vec{c}_j \perp H_j$, for $j \in \{1, 2\}$.

Let $Y_i = \{0, \ldots, n_i\}$ and $Z_i = \{0, n_i\}$.

Let $X = X_1 \times \ldots \times X_k$, where $X_i = Y_i$ if $A_{ii} \leq 0$ and $X_i = Z_i$ if $A_{ii} > 0$.

Definition 3.28. Assume $A_{ii} \leq 0$ and define the map

$$\vec{Z} : X_1 \times \ldots \times X_k \longrightarrow \mathbb{R}^k$$

$$(l_1, \ldots, l_k) \longmapsto \vec{Z}(l_1, \ldots, l_k) = \sum_{i=1}^{k} l_i \vec{e}_i$$

From now on, let us assume that $A_{ij} < 0$, for $i, j \in \{1, \ldots, k\}$.

Definition 3.29. Let $\vec{\ell} \in X$. The $\vec{\ell}$ Nash domain $N(\vec{\ell})$ consists of all points x such that $\vec{\ell}$ is a Nash strategy. The Nash domain N is the union of all $\vec{\ell}$ Nash domains, that is, $N = \cup_{\vec{\ell} \in X} N(\vec{\ell})$.

Our aim is to determine whether $N = \mathbb{R}^k$.

Definition 3.30. The right upper corner from the Nash domain $N(\vec{0})$ is the point $C_{RU}(\vec{0}) = (C_i)_{i \in \{1,\ldots,k\}}$, where

$$C_i = \sum_{j=1}^{n} \alpha_{ij}^N - \alpha_{ii}^N.$$

Let us justify the expression of C_i. If we write the utility of the i individual, choosing N

$$U_i(N) = \sum_{j=1}^{k} \alpha_{ij}^N n_j - \alpha_{ii}^N + \omega_i^N$$

and the utility that the individual will receive switching to Y:

$$U_i(Y) = \omega_i^Y.$$

The condition $U_i(N) \leq U_i(Y)$ gives us the inequality

$$C_i \equiv \sum_{j=1}^{n} \alpha_{ij}^N - \alpha_{ii}^N \geq X_i.$$

Definition 3.31. The right upper corner from the Nash domain $N(\vec{\ell})$ is the point $C_{RU}(\vec{\ell}) = (C_i(\vec{\ell}))_{i \in \{1,\ldots,k\}}$, where

$$C_i(\vec{\ell}) = C_i - \sum_{j=1}^{k} A_{ij} l_j.$$

Let us now justify the expression of $C_i(\vec{\ell})$. Write

$$U_i(Y) = \sum_{j=1}^{k} \alpha_{ij}^{Y} l_j - \alpha_{ii}^{Y} + \omega_i^{Y}$$

and let

$$U_i(Y \to N) = \sum_{j=1}^{k} \alpha_{ij}^{N} (n_j - l_j) + \omega_i^{N}$$

be the utility that i obtains changing to N. Recalling that $A_{ij} = \alpha_{ij}^{Y} + \alpha_{ij}^{N}$, condition $U_i(Y) \geq U_i(Y \to N)$ is equivalent to

$$\sum_{j=1}^{k} A_{ij} l_j - \sum_{j=1}^{k} \alpha_{ij}^{N} n_j - \alpha_{ii}^{Y} \geq -X_i,$$

that is,

$$\sum_{j=1}^{k} \alpha_{ij}^{N} n_j + \alpha_{ii}^{Y} - \sum_{j=1}^{k} A_{ij} l_j \leq X_i,$$

or, equivalently,

$$C_i + \alpha_{ii}^{Y} + \alpha_{ii}^{N} - \sum_{j=1}^{k} A_{ij} l_j \leq X_i.$$

We can write this as

$$C_i + A_{ii} - \sum_{j=1}^{k} A_{ij} l_j \leq X_i.$$

Analogously, write $U_i(N) = \sum_{j=1}^{k} \alpha_{ij}^{N} (n_j - l_j) - \alpha_{ii}^{N} + \omega_i^{N}$ and

$$U_i(N \to Y) = \sum_{j=1}^{k} \alpha_{ij}^{N} (n_j - l_j) + \omega_i^{N}.$$

Condition $U_i(N) \leq U_i(N \to Y)$ is equivalent to

$$\sum_{j=1}^{k} \alpha_{ij}^{N} n_j - \alpha_{ii}^{N} - \sum_{j=1}^{k} A_{ij} l_j \geq X_i.$$

Then, we obtain the conditions that X_i must satisfy to be in $N(\vec{\ell})$:

$$A_{ii} + C_i - \sum_{j=1}^{k} A_{ij} l_j \leq X_i \leq C_i - \sum_{j=1}^{k} A_{ij} l_j.$$

Thus, the right upper corner of the Nash domain $N(\vec{\ell})$ is given by the last definition.

Definition 3.32. We define the lower left corner of a Nash domain $N(\vec{\ell})$ as

$$C_{LoL}(\vec{\ell}) = C_{UR}(\vec{\ell}) - \sum_{i=1}^{k} |A_{ii}|\hat{c}_i.$$

Definition 3.33. For all $\vec{\ell} = (l_1, \ldots, l_k) \in X$, we define

$$N(l_1, \ldots, l_k) = \left\{ C + \vec{Z}(l_1, \ldots, l_k) + w/\ w \in Q(l_1, \ldots, l_k) \right\}, \tag{3.3}$$

where $C = C_{RU}(\vec{0})$ is the right upper corner of $N(\vec{0})$ and $w = (w_1, \ldots, w_k) \in Q(l_1, \ldots, l_k)$ if:

a) $w_i \in [A_{ii}, 0]$, for $l_i \notin \{0, n_i\}$.

b) $w_i \in (-\infty, 0]$, for $l_i = 0$.

c) $w_i \in [-|A_{ii}|, +\infty)$, for $l_i = n_i$.

If $(l_1, \ldots, l_k) \notin X$, then there is no $z = (z_1, \ldots, z_k) \in \mathbb{R}^k$ such that $(l_1, \ldots, l_k)$ is Nash equilibrium, i.e., $N(l_1, \ldots, l_k) = \emptyset$.

Lemma 3.34. *Let $l_i \in X_i$. Then the Nash domain $N(l_1, \ldots, l_k)$ is given by* (3.3)*.*

Proof. We shall separate the proof in three parts. Let $l_i \in \{1, \ldots, n_i - 1\}$, for all i. The $(l_1, \ldots, l_k)$ strategy is a Nash equilibrium if

$$U(i; Y; l_1, \ldots, l_k) \geq U(i; N; l_1, \ldots, l_i - 1, \ldots, l_k)$$

and

$$U(i; N; l_1, \ldots, l_k) \geq U(i; Y; l_1, \ldots, l_i + 1, \ldots, l_k),$$

for all individuals i. If we rearrange all these inequalities, we obtain the expression given in a).

Now, let $l_i = 0$, for some i. In that case, $(l_1, \ldots, l_k)$ is a Nash equilibrium if

$$U(i; N; l_1, \ldots, l_k) \geq U(i; Y; l_1, \ldots, l_i + 1, \ldots, l_k),$$

and

$$U(j; Y; l_1, \ldots, l_k) \geq U(j; N; l_1, \ldots, l_j - 1, \ldots, l_k)$$

and

$$U(j;N;l_1,\ldots,l_k) \geq U(j;Y;l_1,\ldots,l_j+1,\ldots,l_k),$$

for all individuals $j \neq i$. Rearranging all these inequalities, we obtain the expression given in b).

Let $l_i = n_i$, for some i. In this case, $(l_1,\ldots,l_k)$ is a Nash equilibrium if

$$U(i;Y;l_1,\ldots,l_k) \geq U(i;N;l_1,\ldots,l_i-1,\ldots,l_k),$$

and

$$U(j;Y;l_1,\ldots,l_k) \geq U(j;N;l_1,\ldots,l_j-1,\ldots,l_k)$$

and

$$U(j;N;l_1,\ldots,l_k) \geq U(j;Y;l_1,\ldots,l_j+1,\ldots,l_k),$$

for all individuals $j \neq i$. Rearranging these inequalities, gives us the expression c). □

We will separate the cases 2×2 and $k \times 2$, with $k > 2$, as the 2×2 admits an easier geometrical interpretation and allows us to understand better the $k > 2$ situation.

Let us start with the 2×2 case. Recall that $\vec{c_i} = -|A_{ii}|\hat{c}_i$, where $\hat{c}_i$ are the vectors from the canonical basis of $\mathbb{R}^2$.

Definition 3.35. The centre of Nash domain $N(\vec{\ell})$, denoted by $C_N(\vec{\ell})$, is defined as

$$C_N = C_N(\vec{\ell}) = \left\{ C_{RU}(\vec{0}) + \vec{Z}(\vec{\ell}) - \sum_{i=1}^{2} \frac{|A_{ii}|}{2}\hat{c}_i \right\},$$

where $C_{RU}(\vec{0})$ is the right-upper corner of $N(\vec{0})$.

The geometrical situation is shown in Figure 3.1.

Definition 3.36. Let us define the diamond whose vertices are the centres of the domains $N(\vec{\ell}), N(\vec{\ell}+\hat{c}_j), N(\vec{\ell}+\hat{c}_i)$ and $N(\vec{\ell}+\hat{c}_j+\hat{c}_i)$, as

$$L(\vec{\ell}) = \left\{ C_N(\vec{\ell}) + \sum_{i=1}^{2} x_i e_i / \ x_i \in [0,1] \right\}.$$

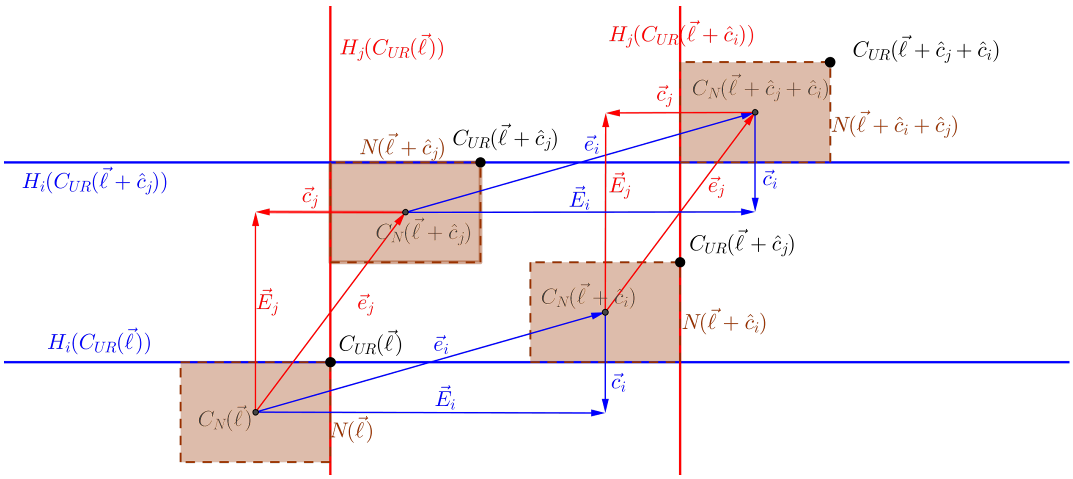

Figure 3.1: Geometrical interpretation of the introduced concepts. Red colour corresponds to direction j and blue color to direction i.

The centre of the diamond is defined by

$$C_L = C_L(\vec{\ell}) = \left\{ C_N(\vec{\ell}) + \frac{1}{2}\sum_{i=1}^{2} \vec{e}_i \right\}.$$

(See Figure 3.2)

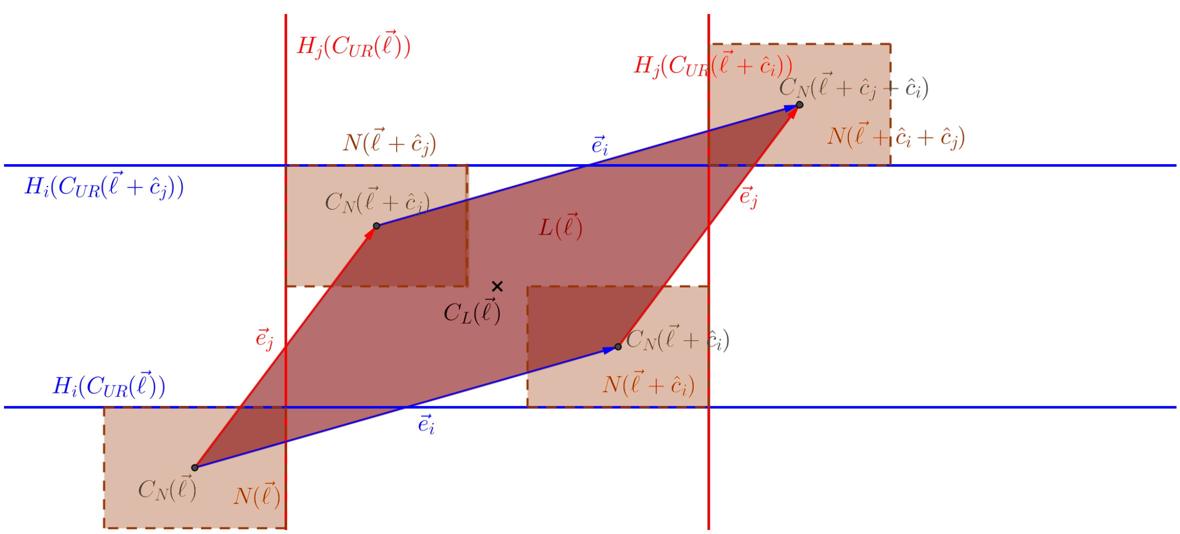

Figure 3.2: Diamond $L(\vec{\ell})$ with its centre, $C_L(\vec{\ell})$.

Remark 3.37. We have that $L(\vec{\ell}) \subset N$ is equivalent to $N = R^2$, as the pattern shown in Figure 3.2 is repeating all over the plane (see Figure 3.3).

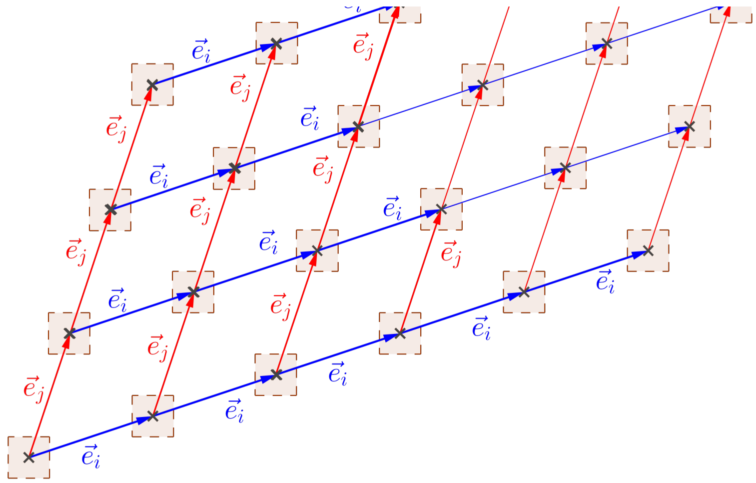

Figure 3.3: The diamond is repeating all over the plane.

Our aim is to cover the diamond $L(\vec{\ell})$ by the Nash domains that are able to cover it. We would like to give equivalent conditions to $L(\vec{\ell}) \subset N$.

We start pointing out that the centre $C_L(\vec{\ell})$ must be the first point to be covered: if the centre is not covered by any domain of the ones appearing in Figure 3.4, then there are some points which are not covered (the centre, at least).

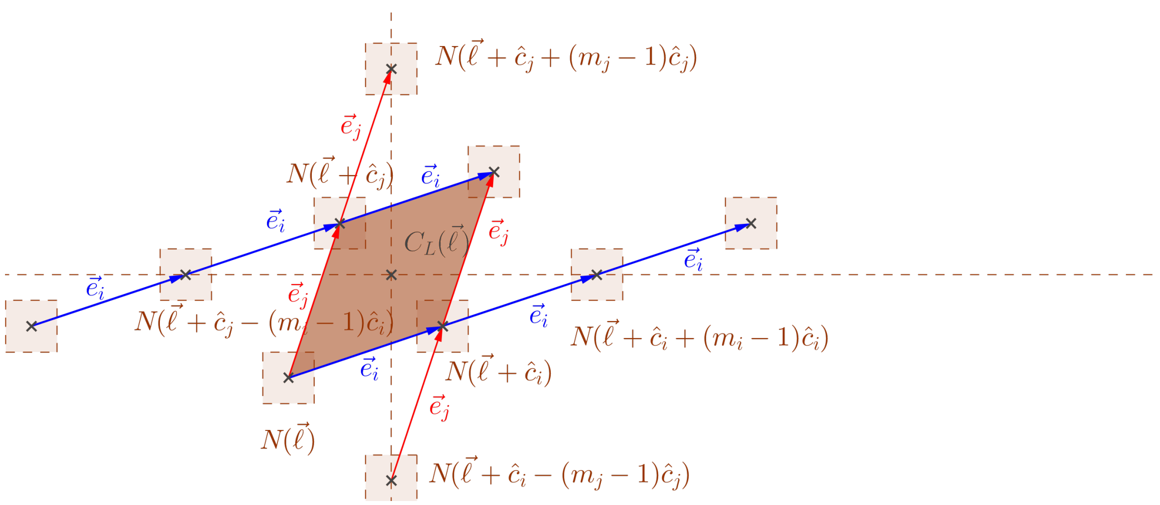

Figure 3.4: Domains that can first cover the centre $C_L(\vec{\ell})$.

Observe that $C_L(\vec{\ell}) \notin N(\vec{\ell}), N(\vec{\ell}+\hat{c}_i+\hat{c}_j)$. In a degenerated case, maybe $C_L(\vec{\ell})$ could be in the boundary of those domains, but in that case it is also on the domains $N(\vec{\ell}+\hat{c}_i)$ and $N(\vec{\ell}+\hat{c}_j)$.

We want to ensure that the centre is covered, although if this condition is verified, there can be some points which are not covered, as we shall see later.

We will try now to give conditions that ensure that any given point $\vec{B} \in L(\vec{\ell})$ is inside of any of the four Nash domains which forms the diamond.

In a general situation, given $\vec{V}$ and $\vec{P}$ in $\mathbb{R}^k$, we would like to know whether $\vec{V}$ is above or below the line (or hyperplane, in general) $H_i(\vec{P})$. For that, let us define $\vec{U} = \vec{V} - \vec{P}$. Observe that $(\vec{U})_i = (\vec{V})_i - (\vec{P})_i$.

In this situation, we can define:

Definition 3.38. We say that

- $\vec{V}$ is upper $H_i(\vec{P})$ when $(\vec{U})_i = (\vec{V})_i - (\vec{P})_i > 0$.
- $\vec{V}$ is lower $H_i(\vec{P})$ when $(\vec{U})_i = (\vec{V})_i - (\vec{P})_i < 0$.
- $\vec{V}$ is on $H_i(\vec{P})$ when $(\vec{U})_i = (\vec{V})_i - (\vec{P})_i = 0$.

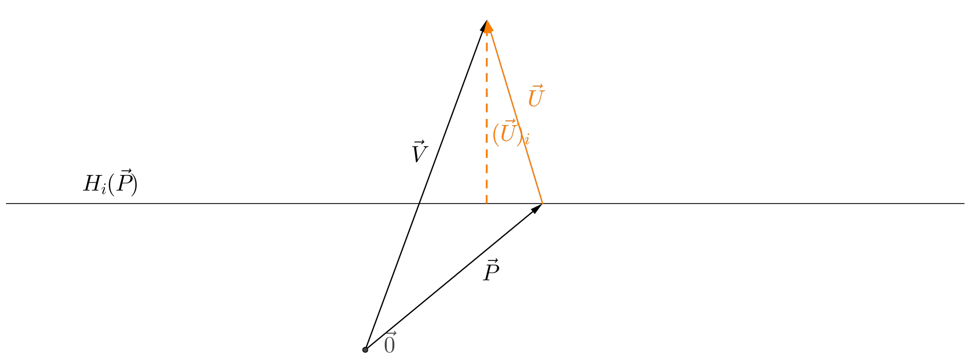

Figure 3.5: $\vec{V}$ is upper $H_i(\vec{P})$

Remark 3.39. If $\vec{Q} \in H_i(\vec{P})$, then $(\vec{Q})_i = (\vec{P})_i$, so $(\vec{V})_i - (\vec{Q})_i = (\vec{V})_i - (\vec{P})_i$.

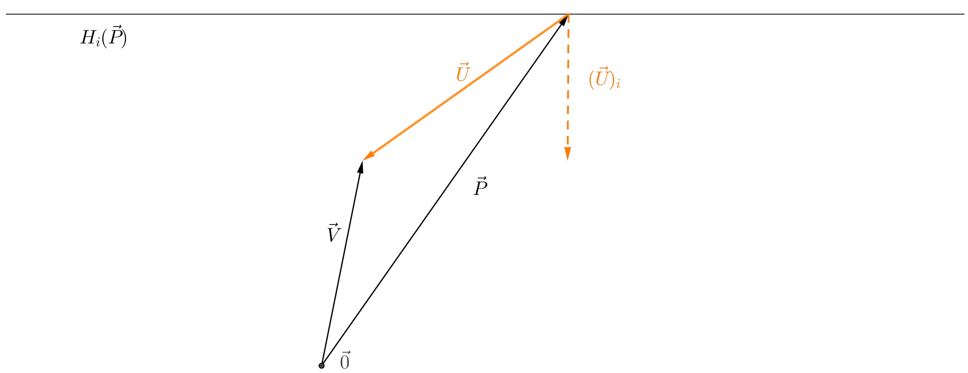

Figure 3.6: $\vec{V}$ is lower $H_i(\vec{P})$

Going back to our case, we can use this definitions to know whether $\vec{B} \in L$ is inside a Nash domain or not. For this, we introduce the following notation.

Notation 13. Given $i \in \{1,2\}$, denote by:

- $\overline{H}_i(\vec{\ell}) = H_i\left(C_N(\vec{\ell}) + \frac{1}{2}\sum_{i=1}^{2}|A_{ii}|\hat{c}_i\right)$.
- $\underline{H}_i(\vec{\ell}) = H_i\left(C_N(\vec{\ell}) - \frac{1}{2}\sum_{i=1}^{2}|A_{ii}|\hat{c}_i\right)$.

Definition 3.40. Given $\vec{B} \in L \subset \mathbb{R}^2$, we have that $\vec{B} \in N(\vec{\ell})$ if and only if $\vec{B}$ is upper $\underline{H}_i(\vec{\ell})$ and lower $\overline{H}_i(\vec{\ell})$, for all $i \in \{1,2\}$, that is,

- $(\vec{B})_i - \left(C_N(\vec{\ell}) - \frac{1}{2}\sum_{i=1}^{2}|A_{ii}|\hat{c}_i\right)_i > 0,$
- $(\vec{B})_i - \left(C_N(\vec{\ell}) + \frac{1}{2}\sum_{i=1}^{2}|A_{ii}|\hat{c}_i\right)_i < 0,$

for all $i \in \{1,2\}$.

Remark 3.41. In the situation where the vectors can be left or right with respect to a given hyperplane, we shall talk about lower and upper when the vector is respectively, to the left and to the right with respect to the given hyperplane, respectively.

Of high interest will be the situation when $\vec{B} \equiv C_L(\vec{\ell})$. In this situation, the two domains $N(\vec{\ell} + \hat{c}_j)$ and $N(\vec{\ell} + \hat{c}_i)$ will cover the centre at the same, for, as we shall see now, whenever $C_L(\vec{\ell}) \in N(\vec{\ell} + \hat{c}_j)$, then also $C_L(\vec{\ell}) \in N(\vec{\ell} + \hat{c}_i)$ and vice-versa.

Lemma 3.42. *Let $A_{ii} < 0$. The following statements are equivalent:*

(1) $C_L(\vec{\ell})$ is upper $\underline{H}_i(\vec{\ell} + \hat{c}_j) = H_i(\vec{e}_j + \frac{\vec{c}_i}{2})$.

(2) $C_L(\vec{\ell})$ is lower $\overline{H}_i(\vec{\ell} + \hat{c}_i) = H_i(\vec{e}_i - \frac{\vec{c}_i}{2})$.

Proof. By its definition, (1) is equivalent to

$$\vec{B}_i - \left(\vec{e}_j + \frac{\vec{c}_i}{2}\right)_i > 0,$$

which is equivalent to

$$\frac{-1}{2}(A_{ij} + A_{ii}) - (-A_{ij} - A_{ii}/2) > 0,$$

which is also equivalent to

$$\frac{A_{ij}}{2} > 0.$$

In the same way, (2) is equivalent to say

$$\vec{B}_i - \left(\vec{e}_i - \frac{\vec{c}_i}{2}\right)_i < 0,$$

which is equivalent to

$$\frac{-1}{2}(A_{ij} + A_{ii}) - (-A_{ii} + A_{ii}/2) < 0,$$

which is also equivalent to

$$-\frac{A_{ij}}{2} < 0.$$

□

Remark 3.43. In a similar way, when $A_{jj} < 0$, one can prove the equivalence of

(1) $C_L(\vec{\ell})$ is upper $\underline{H}_j(\vec{\ell} + \hat{c}_i) = H_j(\vec{e}_i + \frac{\vec{c}_j}{2})$.

(2) $C_L(\vec{\ell})$ is lower $\overline{H}_j(\vec{\ell} + \hat{c}_j) = H_j(\vec{e}_j - \frac{\vec{c}_j}{2})$.

As a consequence of this fact, we obtain the following.

Corollary 3.44. *In the case that $C_L(\vec{\ell}) \in N(\vec{\ell} + \hat{c}_j)$, then, also $C_L(\vec{\ell}) \in N(\vec{\ell} + \hat{c}_i)$ and vice-versa, that is, $C_L(\vec{\ell}) \in N(\vec{\ell} + \hat{c}_j) \cap N(\vec{\ell} + \hat{c}_i)$.*

To check if $C_L(\vec{\ell}) \in N(\vec{\ell} + \hat{c}_j) \cap N(\vec{\ell} + \hat{c}_i)$, one should verify that the four conditions from **Definition 3.40** are met. Nevertheless, due to the geometry of the rectangles, we can simplify this work, in the sense of the next remark.

Remark 3.45. We have the following, because of the geometry of the diamond:

(1) $C_L(\vec{\ell})$ is lower $\overline{H}_i(\vec{\ell} + \hat{c}_j) = H_i(\vec{e}_j - \frac{\vec{c}_i}{2})$.

(2) $C_L(\vec{\ell})$ is upper $\underline{H}_j(\vec{\ell} + \hat{c}_j) = H_j(\vec{e}_j + \frac{\vec{c}_j}{2})$.

As a consequence, we have only to check two of the four conditions from **Definition 3.40** to know whether $C_L(\vec{\ell}) \in N(\vec{\ell} + \hat{c}_j) \cap N(\vec{\ell} + \hat{c}_i)$.

Lemma 3.46. *The following statements are equivalent*

(1) $C_N(\vec{\ell}) + \frac{1}{2}\vec{e}_j \in N(\vec{\ell}) \cup N(\vec{\ell} + \hat{c}_j)$.

(2) $C_N(\vec{\ell}) + \frac{1}{2}x\vec{e}_j \in N(\vec{\ell}) \cup N(\vec{\ell} + \hat{c}_j)$, *for all* $x \in [0, 1]$.

Proof. $(2) \Rightarrow (1)$ is true, taking $x = \frac{1}{2}$.

$(1) \Rightarrow (2)$. Assume that the point $C_N(\vec{\ell}) + \frac{1}{2}\vec{e}_j$ is covered by the two Nash domains. Let us see that the points $C_N(\vec{\ell}) + \frac{1}{2}x\vec{e}_j \in N(\vec{\ell})$, with $x \in [0, \frac{1}{2})$ are lower $\overline{H}_i(\vec{\ell})$. Given $x \in [0, \frac{1}{2})$, the fact that $C_N(\vec{\ell}) + \frac{1}{2}x\vec{e}_j$ is lower $\overline{H}_i(\vec{\ell})$ is equivalent, by definition, to

$$\left(C_N(\vec{\ell}) + \frac{1}{2}x\vec{e}_j\right)_i - \left(C_N(\vec{\ell}) + \frac{1}{2}\sum_{i=1}^{2}|A_{ii}|\hat{c}_i\right)_i < 0,$$

which can be rewritten as

$$\left(\frac{1}{2}x\vec{e}_j - \frac{1}{2}\sum_{i=1}^{2}|A_{ii}|\hat{c}_i\right)_i < 0, \tag{3.4}$$

Let us now prove (3.4). By hypothesis $C_N(\vec{\ell}) + \frac{1}{2}\vec{e}_j \in N(\vec{\ell})$, which, by **Definition 3.40**, gives us that is below $\overline{H}_i(\vec{\ell})$, that is,

$$\left(\frac{1}{2}x\vec{e}_j\right)_i - \left(C_N(\vec{\ell}) + \frac{1}{2}\sum_{i=1}^{2}|A_{ii}|\hat{c}_i\right)_i < 0,$$

condition which can be written as

$$\left(\frac{1}{2}\vec{e}_j - \frac{1}{2}\sum_{i=1}^{2}|A_{ii}|\hat{c}_i\right)_i < 0.$$

This last inequality implies (3.4), as $\frac{1}{2}x\vec{e}_j < \frac{1}{2}\vec{e}_j$.

It can be similarly proved that the points $C_N(\vec{\ell}) + \frac{1}{2}x\vec{e}_j \in N(\vec{\ell})$, with $x \in \left(\frac{1}{2}, 1\right]$ are upper $\underline{H}_i(\vec{\ell} + \hat{c}_j)$, which concludes the proof.

$\square$

Definition 3.47. For $l_i \in X_i$, define, $1 \leq m_i \leq l_i$ as

$$m_i = \left\lceil \frac{|A_{ij}|}{2|A_{ii}|} \right\rceil,$$

where $\lceil\cdot\rceil$ denotes the ceiling function, that is, a function such that $\lceil x \rceil = y \in \mathbb{N}$, where $y - 1 < x \leq y$.

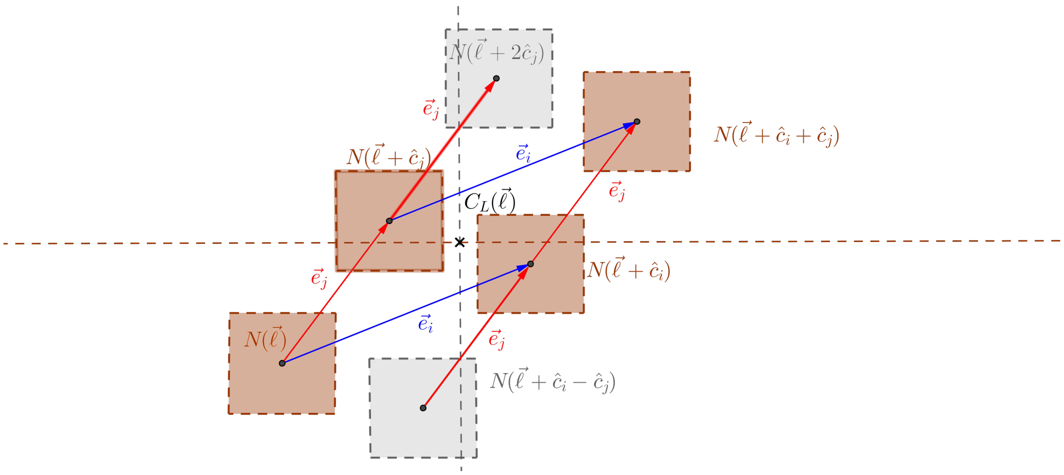

Figure 3.7: Example with $m_j = 2$ and $m_i = 1$.

We will separate our discussion in the cases $m_i = m_j = 1$ and $m_i \geq 2$, for some $i \in \{1, 2\}$. Let us start with $m_i \geq 2$. We start by stating this lemma, which just tells us that, in fact, the domains obtained by m_i, are in a good position to cover the centre of the diamond, $C_L(\vec{\ell})$.

Lemma 3.48. *Let* $m_i \geq 2$*. Then,*

- C_L *is upper* $\underline{H}_i(\vec{\ell} + \hat{c}_j - (m_i - 1)\hat{c}_i)$.
- C_L *is lower* $\overline{H}_i(\vec{\ell} + \hat{c}_j - (m_i - 1)\hat{c}_i)$.
- C_L *is upper* $\underline{H}_i(\vec{\ell} + \hat{c}_i + (m_i - 1)\hat{c}_i)$.
- C_L *is lower* $\overline{H}_i(\vec{\ell} + \hat{c}_i + (m_i - 1)\hat{c}_i)$.

Proof. See figures 3.8 and 3.9. □

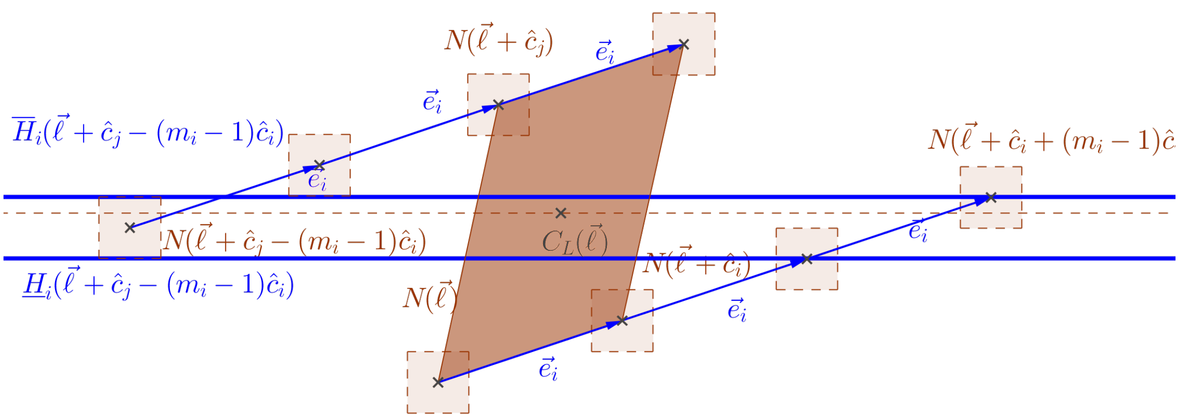

Figure 3.8: Geometric proof of lemma 3.48

Now, we distinguish two cases, depending on if the centre of $N(\vec{\ell} + \hat{c}_i + (m_i - 1)\hat{c}_i)$ is upper or lower $(C_L(\vec{\ell}))_i$.

(1) $m_i A_{ii} \geq (C_L(\vec{\ell}))_i$.

(2) $m_i A_{ii} < (C_L(\vec{\ell}))_i$.

In case (1), it is necessary and sufficient to ask for the domains to reach the centre of the diamond, $C_L(\vec{\ell})$ In case (2), they must cover all of the diamond, as we shall explain now.

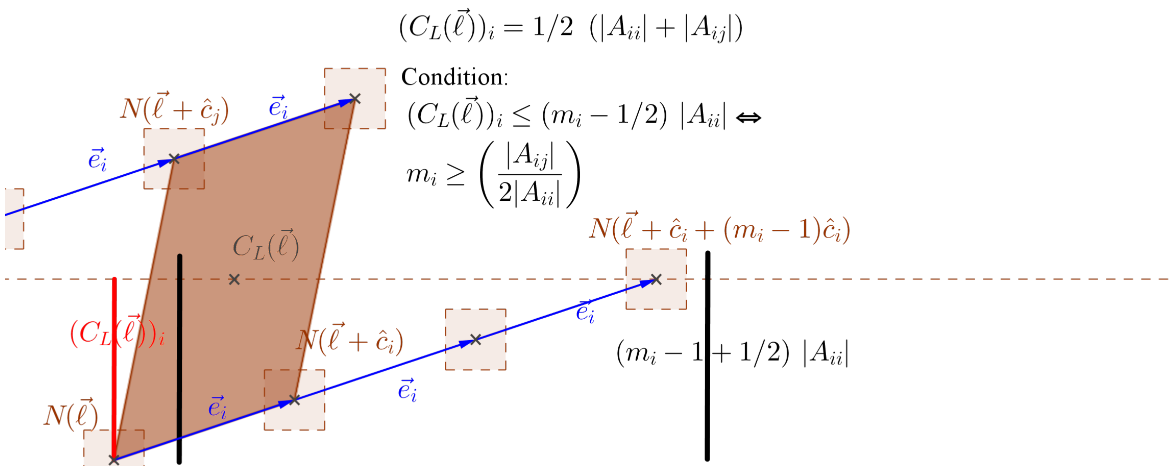

Figure 3.9: interpretation of m_i.

Let us rewrite this cases, using the explicit expression of $(C_L(\vec{\ell}))_i$.

By definition, $C_L(\vec{\ell}) = \frac{1}{2}(\vec{e}_i + \vec{e}_j)$. As, $(\vec{e}_i + \vec{e}_j) = (A_{ji} + A_{jj})\hat{c}_j + (A_{ii} + A_{ij})\hat{c}_i$, then $(C_L(\vec{\ell}))_i = \frac{1}{2}(|A_{ii}| + |A_{ij}|)$.

Then, the two cases are

(1) $m_i A_{ii} \geq \frac{1}{2}(|A_{ii}| + |A_{ij}|) \Leftrightarrow m_i \geq \frac{1}{2} + \frac{|A_{ij}|}{2|A_{ii}|}$.

(2) $m_i A_{ii} < \frac{1}{2}(|A_{ii}| + |A_{ij}|) \Leftrightarrow m_i < \frac{1}{2} + \frac{|A_{ij}|}{2|A_{ii}|}$.

Let us start with the discussion for the case (1). We begin with two remarks.

Remark 3.49. We have that

- $\underline{H}_i(\vec{\ell} + \hat{c}_j - (\hat{m}_i - 1)\hat{c}_i) = \overline{H}_i(\vec{\ell} + \hat{c}_j - \hat{m}_i\hat{c}_i)$, for $1 \leq \hat{m}_i \leq m_i$.
- $\underline{H}_i(\vec{\ell} + \hat{c}_i + \hat{m}_i\hat{c}_i) = \overline{H}_i(\vec{\ell} + \hat{c}_i + (\hat{m}_i - 1)\hat{c}_i)$, for $1 \leq \hat{m}_i \leq m_i$.

Remark 3.50. We have that

$$L \subset \bigcup_{\hat{m}_i=1}^{m_i} \underline{\overline{H}}_i(\vec{\ell} + \hat{c}_j - (\hat{m}_i - 1)\hat{c}_i),$$

where $\underline{\overline{H}}_i(\vec{\ell}+\hat{c}_j-(\hat{m}_i-1)\hat{c}_i)$ denotes the points which are upper $\underline{H}_i(\vec{\ell}+\hat{c}_j-(\hat{m}_i-1)\hat{c}_i)$ and lower $\overline{H}_i(\vec{\ell} + \hat{c}_j - (\hat{m}_i - 1)\hat{c}_i)$.

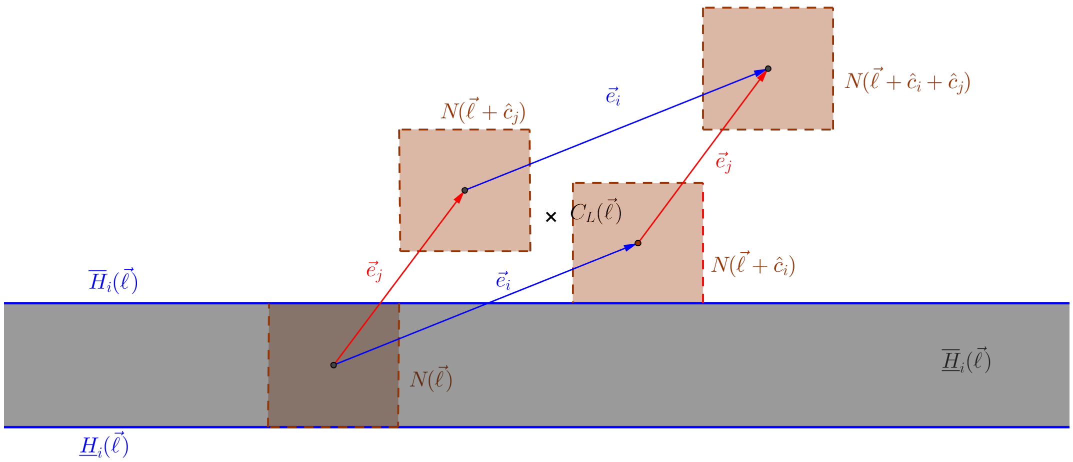

Figure 3.10: Definition of $\underline{\overline{H}}_i(\vec{\ell})$.

Theorem 3.51. *Let* $m_i \geq 2$ *and* $m_i \geq \frac{1}{2} + \frac{|A_{ij}|}{2|A_{ii}|}$. *If*

$$|A_{jj}| \geq \left(m_i - \frac{1}{2}\right)|A_{ji}|,$$

then $C_L \in N(\vec{\ell}+\hat{c}_j-(m_i-1)\hat{c}_i) \cap N(\vec{\ell}+m_i\hat{c}_i)$. *In that case,* $N = \mathbb{R}^2$.

Proof. It follows using **Remark 3.49** and **Remark 3.50** and the following:

If $N(\vec{\ell}+\hat{c}_j-\hat{m}_i\hat{c}_i) \cap L \neq \emptyset$, then

$$N(\vec{\ell}+\hat{c}_j-(\hat{m}_i-1)\hat{c}_i) \cap L = L \cap (\underline{\overline{H}}_i(\vec{\ell}+\hat{c}_j-(\hat{m}_i-1)\hat{c}_i)).$$

Analogously, if $N(\vec{\ell}+\hat{c}_i+\hat{m}_i\hat{c}_i) \cap L \neq \emptyset$, then

$$N(\vec{\ell}+\hat{c}_i+(\hat{m}_i-1)\hat{c}_i) \cap L = L \cap (\underline{\overline{H}}_i(\vec{\ell}+\hat{c}_i+(\hat{m}_i-1)\hat{c}_i)).$$

As the domains $N(\vec{\ell}+\hat{c}_j-(\hat{m}_i-1)\hat{c}_i)$ cover the upper half of the diamond, while the domains $N(\vec{\ell}+\hat{c}_i+(\hat{m}_i-1)\hat{c}_i)$ cover the lower half, then $L \subset N$ and therefore $N = \mathbb{R}^2$.

□

Let us now pass to the case (2). Let us think that $m_i \geq 2$ and thus, the rectangles have to grow on the direction of $\vec{c}_i$. We have to ask the domain which is in the height

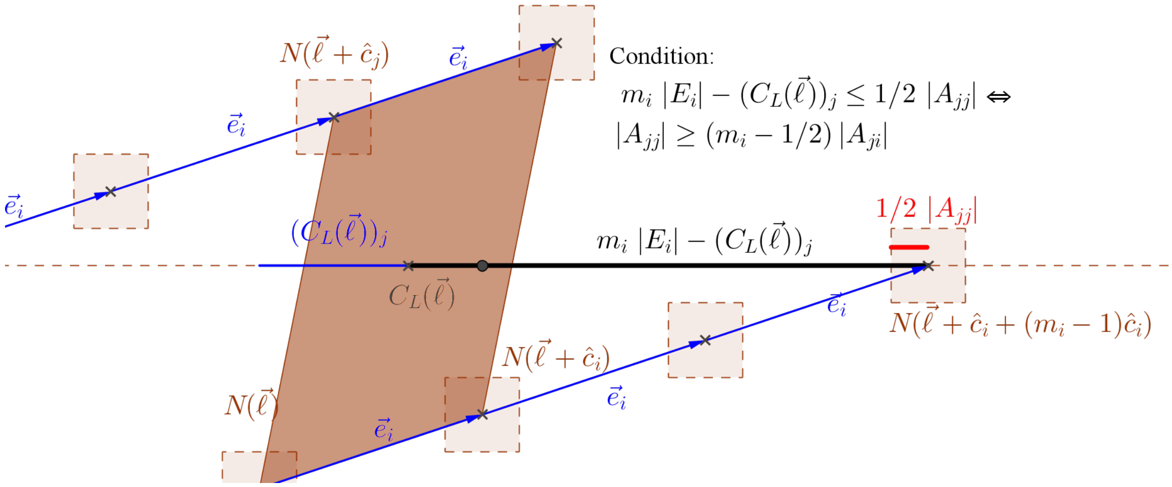

Figure 3.11: Case (1).

of the centre of the diamond, to cross until a point $\vec{v}_j \in C_N(\vec{\ell})$, which is given by the condition $(\vec{v}_j)_i = \left(m_i - \frac{1}{2}\right)|A_{ii}|$ (see Figure 3.12).

As $\vec{e}_j = (|A_{jj}|, |A_{ij}|)$, we obtain that

$$\vec{v}_j = \frac{\left(m_i - \frac{1}{2}\right)|A_{ii}|}{|A_{ij}|}\vec{e}_j.$$

Now, let us obtain the expression of $(\vec{v}_j)_j$. We have that

$$(\vec{v}_j)_j = \left(m_i - \frac{1}{2}\right)|A_{ii}|\frac{|A_{jj}|}{|A_{ij}|}.$$

In this case, the condition to cross $(\vec{v}_j)_j$ is given by

$$m_i|A_{ii}| - (\vec{v}_j)_j \leq \frac{1}{2}|A_{jj}|,$$

which can be rewritten as

$$m_i|A_{ii}| \leq |A_{jj}|\left(\frac{1}{2} + \left(m_i - \frac{1}{2}\right)\frac{|A_{ii}|}{|A_{ij}|}\right).$$

We can state now the following result.

Theorem 3.52. *Let* $m_i \geq 2$ *and* $m_i < \frac{1}{2} + \frac{|A_{ij}|}{2|A_{ii}|}$. *If*

$$m_i|A_{ii}| \leq |A_{jj}|\left(\frac{1}{2} + \left(m_i - \frac{1}{2}\right)\frac{|A_{ii}|}{|A_{ij}|}\right),$$

then $C_L \in N(\vec{\ell} + \hat{c}_j - (m_i - 1)\hat{c}_i) \cap N(\vec{\ell} + m_i\hat{c}_i)$. *In that case,* $N = \mathbb{R}^2$.

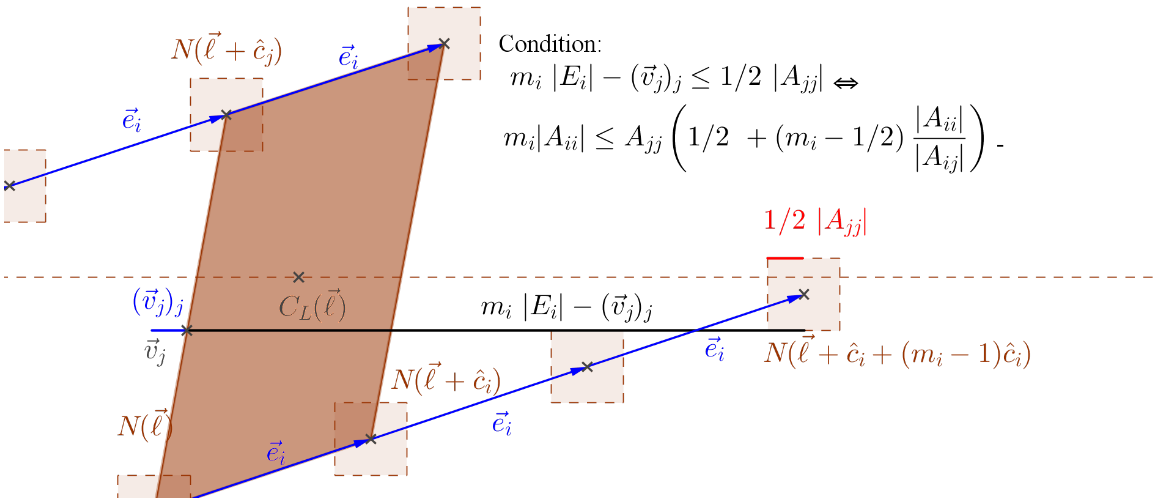

Figure 3.12: Caso (2)

Now let us shift to the case $m_i = m_j = 1$. In this case, the point C_L is inside N, more precisely, in $N(\vec{\ell} + \hat{c}_j) \cap N(\vec{\ell} + \hat{c}_i)$. But there can be points from the diamond which are outside N.

Notation 14. Denote by $N_i(\vec{\ell}) = N(\vec{\ell}) \cup N(\vec{\ell} + \hat{c}_i)$ and $N_j(\vec{\ell}) = N(\vec{\ell}) \cup N(\vec{\ell} + \hat{c}_j)$.

We will distinguish between three different cases, depending on if $C_N(\vec{\ell}) + \frac{1}{2}\vec{e}_i$ and $C_N(\vec{\ell}) + \frac{1}{2}\vec{e}_j$ are inside $N_i(\vec{\ell})$ and $N_j(\vec{\ell})$, respectively, or not. We observe that $C_N(\vec{\ell}) + \frac{1}{2}\vec{e}_i \in N_i(\vec{\ell})$ if and only if $|A_{jj}| \geq |A_{ji}|$, and, similarly, $C_N(\vec{\ell}) + \frac{1}{2}\vec{e}_j \in N_j(\vec{\ell})$ if and only if $|A_{ii}| \geq |A_{ij}|$.

In the first case, consider that $C_N(\vec{\ell}) + \frac{1}{2}\vec{e}_i \notin N_i(\vec{\ell})$ and $C_N(\vec{\ell}) + \frac{1}{2}\vec{e}_j \notin N_j(\vec{\ell})$. As this half-points are not covered, there exists points from the diamond $L(\vec{\ell})$ which are not inside N.

In the second case, consider $C_N(\vec{\ell}) + \frac{1}{2}\vec{e}_i \in N_i(\vec{\ell})$ and $C_N(\vec{\ell}) + \frac{1}{2}\vec{e}_j \in N_j(\vec{\ell})$. In this case, we conclude that $L(\vec{\ell}) \subset N$.

In the last case, we suppose that $C_N(\vec{\ell}) + \frac{1}{2}\vec{e}_i \in N_i(\vec{\ell}), C_N(\vec{\ell}) + \frac{1}{2}\vec{e}_j \notin N_j(\vec{\ell})$ (the case where $C_N(\vec{\ell}) + \frac{1}{2}\vec{e}_j \in N_j(\vec{\ell})$ and $C_N(\vec{\ell}) + \frac{1}{2}\vec{e}_i \notin N_i(\vec{\ell})$ is symmetric to this one). In this case, to ensure that $L(\vec{\ell}) \subset N$, we must make the Nash domains larger in the direction of $\vec{c}_j$. First of all, we notice that $C_N(\vec{\ell}) + \vec{e}_j$ must go across $\overline{H}_i(\vec{\ell}) \cap N(\vec{\ell})$. To justify this fact, recall that, by hypothesis $C_N(\vec{\ell}) + \frac{1}{2}\vec{e}_j \notin N$ and so we have $(\vec{E}_j)_i > A_{ii}$, which implies, $(\vec{E}_j/2)_i > A_{ii}/2$ and $(\vec{e}_j/2)_i > A_{ii}/2$. Recall that $(\vec{e}_j)_i = |A_{ij}|$.

Observe that $C_N(\vec{\ell}) + \frac{|A_{ii}|}{2|A_{ij}|}\vec{e}_j \in \overline{H}_i(\vec{\ell})$. It follows that

$$(C_N(\vec{\ell}))_j \leq \left(C_N(\vec{\ell}) + \frac{|A_{ii}|}{2|A_{ij}|}\vec{e}_j\right)_j \leq (C_{UR}(\vec{\ell}))_j,$$

which means that $C_N(\vec{\ell}) + \frac{|A_{ii}|}{2|A_{ij}|}\vec{e}_j \in \overline{H}_i(\vec{\ell}) \cap N(\vec{\ell})$. If this was not true, then $(\vec{e}_j)_i \neq \left(\vec{E}_j\right)_i$, which contradicts the definition of $\vec{e}_j$ and $\vec{E}_j$.

In a similar way, we can see that $C_N(\vec{\ell}) + \frac{3|A_{ii}|}{2|A_{ij}|}\vec{e}_j \in \overline{H}_i(\vec{\ell} + \hat{c}_j) \cap N(\vec{\ell} + \hat{c}_i + \hat{c}_j)$.

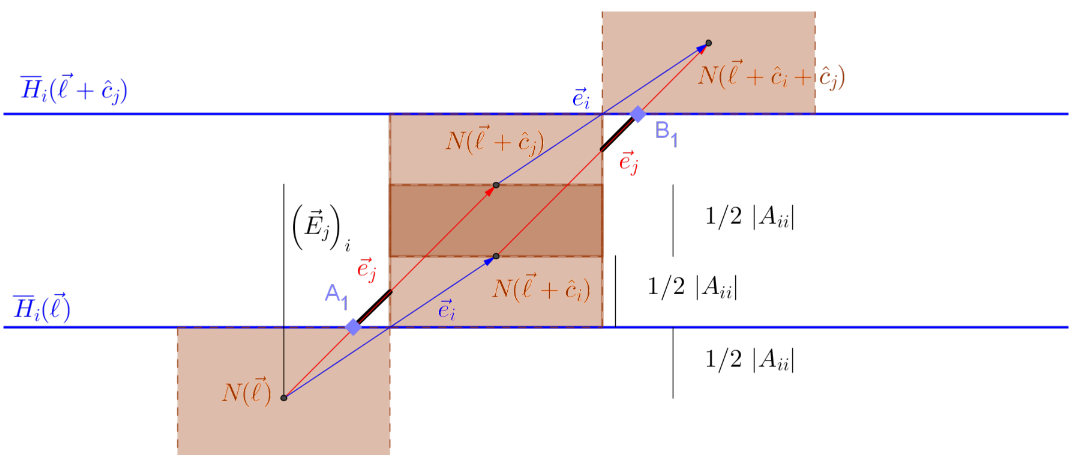

Figure 3.13: Geometrical situation on the third case. The point A_1 corresponds to $C_N(\vec{\ell}) + \frac{|A_{ii}|}{2|A_{ij}|}\vec{e}_j$, while B_1 corresponds to $C_N(\vec{\ell}) + \frac{3|A_{ii}|}{2|A_{ij}|}\vec{e}_j$. The black segments are the parts of the vectors $\vec{e}_j$ which are initially uncovered.

What we have to ask for $C_N(\vec{\ell}) + x\vec{e}_j$, $x \in [0,1]$ to be covered by N, that is equivalent to $C_N(\vec{\ell}) + \frac{|A_{ii}|}{2|A_{ij}|}\vec{e}_j$ is upper $\underline{H}_j(\vec{\ell} + \hat{c}_i) = H_j(C_{LoL}(\vec{\ell} + \hat{c}_i))$. The point $C_N(\vec{\ell}) + \frac{|A_{ii}|}{2|A_{ij}|}\vec{e}_j$ is upper $\underline{H}_j(\vec{\ell} + \hat{c}_i) = H_j(C_{LoL}(\vec{\ell} + \hat{c}_i))$ if and only if

$$\frac{|A_{ii}||A_{jj}|}{2|A_{ij}|} \geq |A_{ji}| - \frac{1}{2}|A_{jj}|.$$

Equivalently, $|A_{jj}|\left(|A_{ii}| + |A_{ij}|\right) \geq 2|A_{ij}||A_{ji}|$.

Theorem 3.53. *When $m_i = m_j = 1$, the centre of the diamond, C_L, is inside N. In*

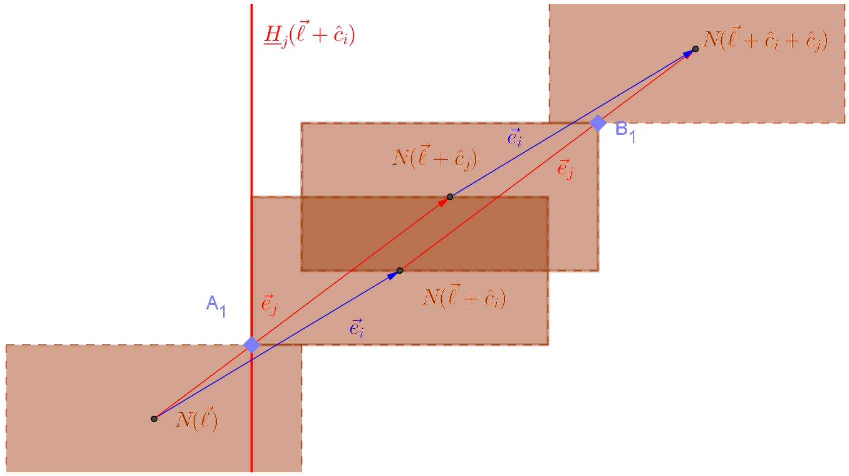

Figure 3.14: Situation when $C_N(\vec{\ell}) + \frac{|A_{ii}|}{2|A_{ij}|}\vec{e}_j$ (represented by A_1) is upper $\underline{H}_j(\vec{\ell}+\hat{c}_i)$

this situation, regarding to points of $C_N(\vec{\ell}) + \frac{1}{2}\vec{e}_i$ and $C_N(\vec{\ell}) + \frac{1}{2}\vec{e}_j$, we have three cases:

(1) $C_N(\vec{\ell}) + \frac{1}{2}\vec{e}_i \notin N_i(\vec{\ell}), C_N(\vec{\ell}) + \frac{1}{2}\vec{e}_j \notin N_j(\vec{\ell})$.

(2) $C_N(\vec{\ell}) + \frac{1}{2}\vec{e}_i \in N_i(\vec{\ell}), C_N(\vec{\ell}) + \frac{1}{2}\vec{e}_j \in N_j(\vec{\ell})$.

(3) $C_N(\vec{\ell}) + \frac{1}{2}\vec{e}_i \in N_i(\vec{\ell}), C_N(\vec{\ell}) + \frac{1}{2}\vec{e}_j \notin N_j(\vec{\ell})$.

In case (1), $N \neq \mathbb{R}^2$. In case (2) $N = \mathbb{R}^2$. In case (3), the following statements are equivalent:

- $N = \mathbb{R}^2$.
- $C_N(\vec{\ell}) + \frac{|A_{ii}|}{2|A_{ij}|}\vec{e}_j$ *is upper* $\underline{H}_j(\vec{\ell}+\hat{c}_i) = H_j(C_{LoL}(\vec{\ell}+\hat{c}_i))$.

Proof. It follows from the previous discussion. □

Putting together all the results obtained in this section, we arrive to the following main theorem.

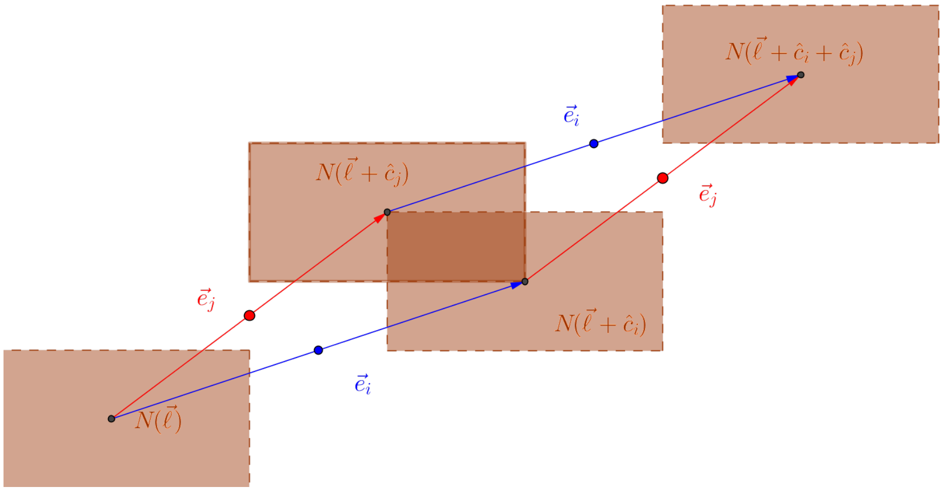

Figure 3.15: Case (1)

Theorem 3.54. *The Nash domain N is equal to $\mathbb{R}^2$ if and only if one of the following conditions holds:*

(i) For $m_i = m_j = 1$,

a) $|A_{jj}| \geq |A_{ji}|$ and $|A_{ii}| \geq |A_{ij}|$; or

b) $|A_{jj}| \geq |A_{ji}|$, $|A_{ii}| < |A_{ij}|$ and $|A_{jj}|\left(|A_{ii}| + |A_{ij}|\right) \geq 2|A_{ij}||A_{ji}|$.

(ii) For $m_i \geq 2$,

a) $2|A_{ii}|m_i \geq |A_{ii}| + |A_{ij}|$ and $2|A_{jj}| \geq (2m_i - 1)|A_{ji}|$; or

b) $2|A_{ii}|m_i < |A_{ii}| + |A_{ij}|$ and $|A_{jj}|(|A_{ij}| + (2m_i - 1)|A_{ii}|) \geq 2m_i|A_{ii}||A_{ij}|$.

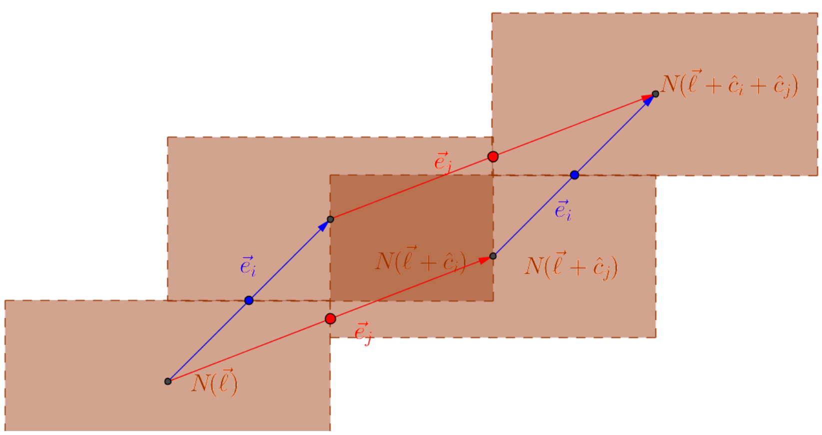

Figure 3.16: Case (2)

Chapter 4

Conclusions

Here, we present a summary of the main results obtained in this thesis and we point out some of the future work lines related with this thesis.

The study of the existence of pure Nash equilibria is a central issue in Game Theory. Here, we carefully studied the shape of Nash domain N for the case of dimension $k = 2$, that determines the existence of pure Nash equilibria. Since we plan to extend our results to the case $k > 2$, we have formalized our results in a way such that they can be extended to the $k > 2$ case.

We have shown that the general decision model is a polymatrix game. Hence, we plan to use the method developed by P. Duarte and his collaborators for the ODE associated to the polymatrix game to prove the existence of chaos for the ODE associated to the general decision model. P. Duarte and his collaborators have shown the existence of chaos for two examples: Lotka-Volterra equation and polymatrix replicator. Since the natural candidates to apply their method are regions without pure Nash equilibria and we have characterized these regions to the general decision model, we have done the first step to study the existence of chaos for the ODE associated to the general decision model.

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